Open questions on effective-dimension proofs of classical theorems: packing dimension and regularity

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Warning

These questions were suggested by Ted Slaman's talk this Monday, so his talk may give a better motivation and intuition

Summary

A set is regular if its Hausdorff and packing dimension coincide

Can we prove Besicovitch-Davies results for analytic regular sets using computability?

Hausdorff dimension

Definition

Let $A \subseteq 2^\omega$, $d \in [0,1]$,

$$\mathcal{H}^d(A) = \lim_{\delta \to 0} \inf \left\{ \sum_i 2^{-|\sigma_i| d} \;\middle|\; \text{there is a cover of } A \text{ by balls } B(\sigma_i) \text{ with } 2^{-|\sigma_i|} < \delta \right\}$$

$$\dim_{\mathrm{H}}(A) = \inf \left\{ d \geq 0 \;\middle|\; \mathcal{H}^d(A) = 0 \right\}$$

Packing dimension

Definition

Let $A \subseteq 2^\omega$, $d \in [0,1]$,

$$\mathcal{P}_0^d(A) = \limsup_{\delta \to 0} \left\{ \sum_i 2^{-|\sigma_i| d} \;\middle|\; \text{there is a packing of } A \text{ by disjoint } B(\sigma_i) \text{ with } 2^{-|\sigma_i|} < \delta \right\}$$

$$\mathcal{P}^d(A) = \inf \left\{ \sum_{U \in \mathcal{U}} \mathcal{P}_0^d(U) \;\middle|\; \mathcal{U} \text{ is a countable cover of } A \right\}$$

$$\dim_{\mathrm{P}}(A) = \inf \left\{ d \geq 0 \;\middle|\; \mathcal{P}^d(A) = 0 \right\}$$

Effectivization

Let $x \in 2^\omega$

Definition

$$\dim(x) = \liminf_n \frac{\mathrm{K}(x \upharpoonright n)}{n},$$

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Definition

Let $A \subseteq 2^\omega$,

$$\dim(A) = \sup_{x \in A} \dim(x).$$

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Point-to-set principles

Theorem (Lutz Lutz 2018)

Let $A \subseteq 2^\omega$. *Then*

$$\dim_{\mathrm{H}}(A) = \min_{B \subseteq \mathbb{N}} \dim^B(A).$$

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Let $A \subseteq 2^\omega$. *Then*

$$\dim_{\mathrm{P}}(A) = \min_{B \subseteq \mathbb{N}} \mathrm{Dim}^B(A).$$

Question 1

Definition

A is regular if $\dim_{\mathrm{H}}(A) = \dim_{\mathrm{P}}(A)$.

Can computability (partially) characterize regularity for $A \subseteq 2^{\omega}$?

Capacitability

Theorem (Besicovitch-Davies / Frostman)

Let A be analytic, $\dim_{\mathrm{H}}(A) = d$. For every $s < d$ there is a closed $C \subseteq A$ with $s \leq \dim_{\mathrm{H}}(C)$.

Theorem (Joyce Preiss)

Let A be analytic, $\dim_{\mathrm{P}}(A) = d$. For every $s < d$ there is a compact $K \subseteq A$ with $s \leq \dim_{\mathrm{P}}(K)$.

Question 2

Theorem

Let $A \subseteq 2^\omega$ be analytic with $\dim_{\mathrm{P}}(A) = \dim_{\mathrm{H}}(A)$. For every $s < d$ there is a closed $C \subseteq A$ with $s \leq \dim_{\mathrm{H}}(C)$.

Can we prove this result using the point-to-set principles?

Extensions

- **Question 3** The same question for other separable spaces
- **Question 4** The same question for other gauge functions/families