Open Question

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Admissible Sets

- **KPU** — axioms of Kripke-Pkatek with urelements.
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Examples

- $\mathbb{HF}(\mathfrak{M})$ and $\mathbb{HF}(\varnothing)$.
- $\mathbb{L}(\alpha)$ where α is admissible.
- $\mathbb{L}(\alpha;\mathfrak{M})$ where α is $\mathfrak{M}$-admissible.
- $\mathbb{HYP}(\mathfrak{M}) = \mathbb{L}(\alpha_0;\mathfrak{M})$ where α_0 is least $\mathfrak{M}$-admissible.

Computability

Σ (possibly with parameters) - c.e.
Δ - c.

Weak Computability in...

Definition

$\mathfrak{M}$ is Σ-definable in $\mathbb{A}$ iff there exists a map ν from $\mathbb{A}$ onto $\mathfrak{M}$ such that ν-preimages of all the signature relations together with the equality relation on $\mathfrak{M}$ are Δ on $\mathbb{A}$.

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Proposition

$\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\varnothing)$ iff $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$ iff $\mathfrak{M}$ is computable.

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$\mathfrak{M}$ is definable in $\mathbb{A}$ iff there exists a map ν from $\mathbb{A}$ onto $\mathfrak{M}$ such that ν-preimages of all the signature relations together with the equality relation on $\mathfrak{M}$ are definable by some formulas on $\mathbb{A}$.

Weak Computability in...

Effective interpretability

Effective interpretability of a structure $\mathfrak{M}$ in $\mathfrak{N}$ coincides with parameterless version of the notion "$\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$", for countable structures $\mathfrak{M}$ and $\mathfrak{N}$
(M. Harrison-Trainor, A. Melnikov, R. Miller, A. Montalban).

Strong Computability

Definition

$\mathbb{A}$ is Σ-reducible to $\mathbb{B}$ (shortly $\mathbb{A} \sqsubseteq_{\Sigma} \mathbb{B}$) iff there exists a map ν from $\mathbb{B}$ onto $\mathbb{A}$ such that $\nu^{-1}(\Sigma(\mathbb{A})) \subseteq \Sigma(\mathbb{B})$ (preserving arity).

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Proposition

$\mathfrak{M}$ is Σ-definable in $\mathbb{A}$ iff $\mathbb{HF}(\mathfrak{M})$ is Σ-definable in $\mathbb{A}$ iff $\mathbb{HF}(\mathfrak{M}) \sqsubseteq_{\Sigma} \mathbb{A}$.

Strong Computability

Theorem

For any admissible set $\mathbb{A}$, there exists a directed graph $\mathfrak{M}_{\mathbb{A}}$ such that $\mathbb{A} \equiv_{\Sigma} \mathbb{HF}(\mathfrak{M}_{\mathbb{A}})$. Moreover, this transformation preserves certain properties of $\mathbb{A}$.

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Corollary

$\mathbb{A} \sqsubseteq_{\Sigma} \mathbb{B}$ iff for each structure $\mathfrak{M}$, if $\mathfrak{M}$ is Σ-definable in $\mathbb{A}$ then $\mathfrak{M}$ is Σ-definable in $\mathbb{B}$.

Jump Structure

Definition

$\mathcal{J}(\mathbb{A}) = (\mathbb{HF}(\mathfrak{M}_{\mathbb{A}}), P)$ where P is a universal Σ-predicate on $\mathbb{HF}(\mathfrak{M}_{\mathbb{A}})$.
$\mathcal{J}^{n+1}(\mathbb{A}) = \mathcal{J}(\mathcal{J}^{n}(\mathbb{A}))$.

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Theorem

$\mathfrak{M}$ is definable in $\mathbb{A}$ iff $\mathfrak{M}$ is Σ-definable in $\mathcal{J}^{n}(\mathbb{A})$ for some $n \in \omega$.

A fixed point

Theorem ($\mathsf{KP} + 0^{\#}$ (A. Montalban, 2013); $\mathsf{KP} + \textit{Power}$ (P, 2011))

For any cardinality α, there exists an admissible set $\mathbb{A}_\alpha$ such that $\mathcal{J}(\mathbb{A}_\alpha) \equiv_{\Sigma} \mathbb{A}_\alpha$.

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Theorem

$\mathbb{A} \equiv_{\Sigma} \mathcal{J}(\mathbb{A})$ iff for any structure $\mathfrak{M}$, if $\mathfrak{M}$ is definable in $\mathbb{A}$ then $\mathfrak{M}$ is Σ-definable in $\mathbb{A}$.

Conjecture

For every admissible set $\mathbb{A}$, the following conditions are equivalent:
1) $\mathbb{A} \equiv_\Sigma \mathcal{J}(\mathbb{A})$;
2) for any admissible set $\mathbb{B}$, if $\mathbb{B}$ is Σ-definable (as a structure) in $\mathbb{A}$ then $\mathbb{B} \sqsubseteq_\Sigma \mathbb{A}$.

(1)⇒(2)

Very simple:
$[\mathbb{B}$ is Σ-definable in $\mathbb{A}] \Longrightarrow$
$[\mathbb{B}$ is definable in $\mathbb{A}] \Longrightarrow$
$[\mathfrak{M}_{\mathbb{B}}$ is definable in $\mathbb{A}] \Longrightarrow$
$[\mathfrak{M}_{\mathbb{B}}$ is Σ-definable in $\mathbb{A}] \Longrightarrow$
$\mathbb{HF}(\mathfrak{M}_{\mathbb{B}}) \sqsubseteq_{\Sigma} \mathbb{A} \Rightarrow \mathbb{B} \sqsubseteq_{\Sigma} \mathbb{A}$.

(2)⇒(1)

Proposition (P,2005)

$\mathbb{HYP}(\mathfrak{N})$ is Σ-definable in $\mathbb{HF}(\mathbb{HYP}(\mathfrak{N}))$ but $\mathbb{HYP}(\mathfrak{N}) \not\subseteq_\Sigma \mathbb{HF}(\mathbb{HYP}(\mathfrak{N}))$.

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Theorem (2019; Avdeev, P)

$\mathbb{HYP}(\mathfrak{M}) \sqsubseteq_\Sigma \mathbb{HF}(\varnothing)$ iff $\mathfrak{M}$ is recursively saturated and is decidable.

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There exists a structure $\mathfrak{M}$ such that $\mathbb{HYP}(\mathfrak{M})$ is computable as a structure but $\mathbb{HYP}(\mathfrak{M}) \not\sqsubseteq_\Sigma \mathbb{HF}(\varnothing)$.

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Relativization

If $\Delta(\mathbb{A}) \cap \mathcal{P}(\omega)$ is not closed under the jump operation then there exists a structure $\mathfrak{M}$ such that $\mathbb{HYP}(\mathfrak{M})$ is Σ-definable in $\mathbb{A}$ but $\mathbb{HYP}(\mathfrak{M}) \not\sqsubseteq_\Sigma \mathbb{A}$.