Oberwolfach Workshop

April 26–May 1, 2021

Computability Theory

Densely Computable Structures

Valentina Harizanov
George Washington University

harizanv@gwu.edu
http://home.gwu.edu/~harizanv/

Joint work with Wesley Calvert and Douglas Cenzer.

Generic case complexity

- It would be worthwhile to distinguish which results in computable model theory depend on "special" (and potentially rare) inputs.

- For problems on groups, Kapovich, Myasnikov, Schupp, and Shpilrain proposed using notions of asymptotic density to see whether a partial computable function could solve "almost all" instances of a problem.

- They showed that for a large class of finitely generated groups the classical decision problems, such as the word problem or the conjugacy problem, have linear time generic case complexity.

- Kapovich, Myasnikov, Schupp, and Shpilrain established that a finitely presented group with undecidable word problem, given by W. Boone, has a generically computable copy.

- Jockusch and Schupp extended this approach to the broader context in computability theory. They introduced and studied *generically computable* and *coarsely computable* sets of natural numbers.

- For $A \subseteq \omega$ and $n \geq 1$, the density of a set A up to n, denoted by $\rho_n(A)$, is
$$\frac{|A \cap \{0, 1, 2, \ldots, n-1\}|}{n}$$

- The *(asymptotic) density* of A is $\rho(A) = \lim_n \rho_n(A)$.

- For example, $A = \{2^n : n \in \omega\}$ has density 0.

- A is (asymptotically) dense if $\rho(A) = 1$.

- The upper density of A is $\limsup_n \frac{|(A \cap n)|}{n}$.

  It is 1 if there is a sequence $n_0 < n_1 < \cdots$ such that $\lim_i \rho_{n_i}(A) = 1$.

- If A is a c.e. set with upper density 1, then A has a computable subset with upper density 1.

Generically and coarsely computable sets (Jockusch and Schupp)

- For $S \subseteq \omega$, let c_S denote the characteristic function of S.

- S is *generically computable* if there is a partial computable function $\varphi : \omega \to \{0,1\}$ such that: $dom(\varphi)$ has asymptotic density 1, and $c_s \upharpoonright dom(\varphi) = \varphi$.

- S is *coarsely computable* if there is a *total* computable function $\tau : \omega \to \{0,1\}$ such that $\{x : c_S(x) = \tau(x)\}$ has asymptotic density 1.

  Equivalently, S is coarsely computable if there is a computable set T such that $S \triangle T$ has asymptotic density 0.

- (Jockusch and Schupp)

  There is a coarsely computable c.e. set that is not generically computable.

  There is a generically computable c.e. set that is not coarsely computable.

- A structure $\mathcal{D}$ for a finite language is c.e. if its domain D is c.e. and each relation of $\mathcal{D}$ is c.e. and each function of $\mathcal{D}$ is the restriction of a partial computable function to D.

Asymptotic density in $\omega \times \omega$

- Let $A \subseteq \omega$. Then A has asymptotic density δ in ω if and only if $A \times A$ has asymptotic density δ^2 in $\omega \times \omega$.

  Hence: A is asymptotically dense in ω iff $A \times A$ is asymptotically dense in $\omega \times \omega$.

- There is a computable dense set $C \subseteq \omega \times \omega$ such that for any infinite c.e. set $E \subseteq \omega$, the product $E \times E$ is not a subset of C.

Generically computable structures

- Consider a structure $\mathcal{A}$ for finite language with universe ω, with functions $\{f_i : i \in I\}$, each f_i of arity p_i, and relations $\{R_j : j \in J\}$, each R_j of arity r_j.

- We call $\mathcal{A}$ *generically computable* if $\mathcal{A}$ has a substructure $\mathcal{D}$ with a c.e. domain D of asymptotic density 1, and partial computable functions $\{\phi_i : i \in I\}$ and $\{\psi_j : j \in J\}$ such that each ϕ_i agrees with f_i on D^{p_i} and each ψ_j agrees with c_{R_j} on the set D^{r_j}.

Example

- Let $\mathcal{M} = (\omega, A)$, where A is a unary relation.

- Assume that A is a generically computable set. Let a partial computable function φ be such that: $dom(\varphi)$ has density 1, and for every $x \in dom(\varphi)$, we have $c_A(x) = \varphi(x)$. Let $D = dom(\varphi)$. Consider the substructure $\mathcal{D} = (D,\ A \cap D)$. Since D is c.e. and φ is $c_{A \cap D}$ on D, the structure $\mathcal{M}$ is generically computable.

- Assume that $\mathcal{M}$ is a generically computable structure with a substructure $\mathcal{D} = (D,\ A \cap D)$ with a dense c.e. domain D such that $c_{A \cap D}$ extends to a partial computable function. Let φ be the restriction of that function to D. The restriction is partial computable and agrees with c_A on D, so the set A is generically computable.

Σ_n generically c.e. structures

- A substructure $\mathcal{B}$ is a Σ_n *elementary substructure* of $\mathcal{A}$ if for any infinitary Σ_n formula $\theta(x_1, \ldots, x_n)$ and $b_1, \ldots, b_n \in \mathcal{B}$:

  $\mathcal{A} \models \theta(b_1, \ldots, b_n)$ iff $\mathcal{B} \models \theta(b_1, \ldots, b_n)$

- A structure $\mathcal{A}$ is Σ_n *generically c.e.* if there is a c.e. dense set D such that the substructure $\mathcal{D}$ with universe D is a c.e. substructure and also a Σ_n elementary substructure of $\mathcal{A}$.

- Clearly, a Σ_{n+1} generically c.e. structure is Σ_n generically c.e.

- A computable structure is Σ_n generically c.e. for any n.

Generically computable injection structures

- An *injection structure* $\mathcal{A} = (A, f)$ has a single unary function f that is $1-1$.

- Any c.e. injection structure is isomorphic to a computable injection structure.

- For $a \in A$, the *orbit* of a is

$$\mathcal{O}_f(a) = \{b \in A : (\exists n \in \mathbb{N})[f^n(a) = b \vee f^n(b) = a]\}$$

- The *character* of $\mathcal{A}$ is defined as:
$\chi(\mathcal{A}) = \{\langle k, n\rangle : n, k > 0 \text{ \& there are } \geq n \text{ orbits of size } k\}.$

- An injection structure $\mathcal{A} = (\omega, f)$ has a *generically computable copy* iff

  (i) $\mathcal{A}$ has an infinite substructure isomorphic to a computable structure iff

  (ii) $\mathcal{A}$ has an infinite orbit or $\chi(\mathcal{A})$ has an infinite c.e. subset.

- $\mathcal{A} = (\omega, f)$ has a Σ_1 *generically c.e.* copy iff

  (i) $\mathcal{A}$ has a computable copy iff

  (ii) $\chi(\mathcal{A})$ is a c.e. set iff

  (iii) $\mathcal{A}$ has a Σ_2 generically c.e. copy.

Computable and c.e. equivalence structures

- For an equivalence structure $\mathcal{A} = (A, E)$:

  The *character* of $\mathcal{A}$ (or E) is defined as:
  $\chi(\mathcal{A}) = \{\langle k, n\rangle : n, k > 0$ & there are $\geq n$ equivalence classes of size $k\}$

- If A and E are c.e., the character $\chi(\mathcal{A})$ is a $\mathbf{\Sigma}^0_2$ set.

- $K \subseteq \langle(\omega - \{0\}) \times (\omega - \{0\})\rangle$ is a *character* if for all $n > 0$ and k:

$$\langle k, n+1\rangle \in K \Rightarrow \langle k, n\rangle \in K$$

- K is a character if $K = \chi(\mathcal{A})$ for some equivalence structure $\mathcal{A}$.

- (Calvert, Cenzer, Harizanov, Morozov)
  For any Σ^0_2 character K, there is a computable equivalence structure $\mathcal{A}$ with character K and infinitely many infinite equivalence classes.

- (Cenzer, Harizanov, Remmel)
  For any Σ^0_2 character K, there is a c.e. equivalence structure, even with a computable domain, with character K and with any finite number $r \geq 1$ of infinite equivalence classes.

Generically computable equivalence structures

- If an equivalence structure $\mathcal{A} = (\omega, E)$ is generically computable, then there is an infinite computable set $Y \subseteq \omega$ such that the restriction of E to $Y \times Y$ is computable.

- Every equivalence structure $\mathcal{A} = (\omega, E)$ has a generically computable copy.

Σ_1 and Σ_2 generically c.e. equivalence structures

- A function $h : \omega^2 \to \omega$ is a (Khisamiev's) s_1-*function* if for all i, t,

  $h(i,t) \leq h(i,t+1)$,

  $m_i = \lim_s h(i,s)$ exists, and

  $m_0 < m_1 < \cdots < m_i < \cdots$

- Let $\mathcal{A} = (A, E)$ be a c.e. equivalence structure with no infinite equivalence classes and an unbounded character. Then there is a computable s_1-function h such that $\mathcal{A}$ contains an equivalence class of size m_i for each $i \in \omega$.

- We say that a character has an s_1-function h if it has an equivalence class of size m_i for each i.

- For every Σ^0_2 character K that is either bounded or has a computable s_1-function, there is a computable equivalence structure $\mathcal{A}$ with character K and no infinite equivalence classes.

- If $\mathcal{A}$ is c.e. equivalence structure with no infinite equivalence classes, then $\mathcal{A}$ is isomorphic to a computable structure.

- An equivalence structure $\mathcal{A} = (\omega, E)$ has a Σ_1 *generically c.e. copy* iff at least one of the following conditions hold:

  1. $\chi(\mathcal{A})$ is bounded;

  2. $\chi(\mathcal{A})$ has a Σ^0_2 subset K, a character with a computable s_1-function;

  3. $\mathcal{A}$ has an infinite class and $\chi(\mathcal{A})$ has a Σ^0_2 subset K;

  4. $\mathcal{A}$ has infinitely many infinite classes.

- $\mathcal{A} = (\omega, E)$ has a Σ_2 *generically c.e. copy* iff

  (i) $\mathcal{A}$ has a c.e. copy iff

  (ii) $\mathcal{A}$ has a Σ_3 generically c.e. copy.

Coarsely computable structures

- A structure $\mathcal{A}$ is *coarsely computable* if there are a computable structure $\mathcal{E}$ and a dense set D such that the structure $\mathcal{D}$ with universe D is a substructure of both $\mathcal{A}$ and of $\mathcal{E}$ and all relations and functions agree on D :

$$\mathcal{D} \subseteq \begin{matrix} \mathcal{A} \\ \mathcal{E} \end{matrix}$$

- $\mathcal{A} = (\omega, A)$ is a coarsely computable structure iff A is a coarsely computable set.

- There is a generically computable structure that is not coarsely computable, and there is a coarsely computable structure that is not generically computable.

Σ_n coarsely c.e. structures

- A structure $\mathcal{A}$ is Σ_n *coarsely c.e.* if there are a c.e. structure $\mathcal{E}$ and a dense set D such that the structure $\mathcal{D}$ with universe D is a Σ_n elementary substructure of both $\mathcal{A}$ and $\mathcal{E}$ and all relations and functions agree on D :

$$\mathcal{D} \preceq_n \begin{matrix} \mathcal{A} \\ \mathcal{E} \end{matrix}$$

- A Σ_0 coarsely c.e. structure is also called a *coarsely c.e.* structure.

- Clearly, a Σ_{n+1} coarsely c.e. structure is Σ_n coarsely c.e.

  A coarsely computable structure is a coarsely c.e. structure.

Coarsely computable and Σ_1 coarsely c.e. injection structures

- Any generically computable injection structure has a coarsely computable copy.

- There is a generically computable injection structure that is not coarsely computable.

- There is a coarsely computable injection structure with no generically computable copy.

- *Proof idea*. Let $K \subseteq \omega - \{0\}$ be a dense immune set.

  Build an injection structure $\mathcal{A}$ with character

  $\{\langle k, i\rangle : k \in K \wedge 1 \leq i \leq 2\}$ and no infinite orbits.

  If $\mathcal{B}$ were a generically computable copy of $\mathcal{A}$, then $\chi(\mathcal{A}) = \chi(\mathcal{B})$ would contain an infinite c.e. subset C.

  Then $\{k : \langle k, 1\rangle \in C \vee \langle k, 2\rangle \in C\}$ is an infinite c.e. subset of K, a contradiction.

- There is an injection structure that does not have a coarsely computable copy.

- *Proof idea*. Build an infinite set $K \subseteq \omega$ such that an injection structure $\mathcal{A}$ with character $\chi(\mathcal{A}) \subseteq K \times \{1\}$ cannot be coarsely computable.

- An injection structure $\mathcal{A} = (\omega, f)$ has a Σ_1 *coarsely c.e. copy* iff

  (i) $\mathcal{A}$ has a computable copy iff

  (ii) $\chi(\mathcal{A})$ is a c.e. set.

Σ_n coarsely c.e. equivalence structures

- There is an equivalence structure with no Σ_1 coarsely c.e. copy.

- There is a Σ_1 coarsely c.e. equivalence structure with no Σ_1 generically c.e. copy.

- For any equivalence structure $\mathcal{A}$,

  $\mathcal{A}$ is Σ_3 *coarsely c.e.* iff

  $\mathcal{A}$ has a c.e. copy.

- Let $\mathcal{A}$ be an equivalence structure with an infinite class, or with a bounded character, or with an unbounded character that has a computable s_1-function.

  Then $\mathcal{A}$ has a Σ_2 *coarsely c.e. copy* iff

  $\chi(\mathcal{A})$ is Σ^0_2 iff

  $\mathcal{A}$ has a c.e. copy.

- Let $\mathcal{A}$ be an equivalence structure with no infinite classes, with an unbounded character with no computable s_1-function.

  Then $\mathcal{A}$ has a Σ_2 *coarsely c.e. copy* iff

  $\chi(\mathcal{A})$ is Σ^0_2 and for some finite k, $\mathcal{A}$ has infinitely many classes of size k.

THANK YOU!