# Coarse computability, the density metric, and Hausdorff distances between Turing degrees

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Joint work with Carl G. Jockusch, Jr. and Paul E. Schupp

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The completeness of $(\mathcal{S}, \delta)$ is equivalent over $RCA_0$ to the principle DOM, studied by Hölzl, Jain, Raghavan, Stephan, and Zhang.

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An example is $\mathcal{U} = \{Y : C \nleqslant_{\mathrm{T}} Y\}$ for a noncomputable $C$.

For $[A] \in \mathcal{S}$ and $\mathcal{B} \subseteq S$, let $\delta([A], \mathcal{B}) = \inf\{\delta([A], [B]) : [B] \in \mathcal{B}\}$.

The *Hausdorff distance* between $\mathcal{A}, \mathcal{B} \subseteq \mathcal{S}$ is
$H(\mathcal{A}, \mathcal{B}) = \max\big(\sup\{\delta([A], \mathcal{B}) : [A] \in \mathcal{A}\}, \sup\{\delta([B], \mathcal{A}) : [B] \in \mathcal{B}\}\big)$.

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We can understand this space better via the relativized form of the *coarse computability bound*.

$$\gamma(A) = \sup\{r : (\exists \text{ computable } C)[\delta(A, C) \leqslant 1 - r]\}.$$

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Let *a.e.* $\equiv$ "for almost every" and *c.m.* $\equiv$ "for comeager many".

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$\mathbf{a}$ is *attractive* if $(\text{a.e.}\,\mathbf{b})[H(\mathbf{a},\mathbf{b}) = \frac{1}{2}]$, and *dispersive* otherwise.

For example, 1-randoms are attractive, by van Lambalgen's Theorem.

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The last theorem cannot be improved to weak 2-randomness, but we do know whether it can be improved to 1-genericity.

**Open Question.** Is every 1-generic degree dispersive?

Let $\mathcal{M}$ be a $(0, \frac{1}{2}, 1)$-valued metric space.

Let $G_{\mathcal{M}}$ be the graph with vertices the points in $\mathcal{M}$, and an edge between $x$ and $y$ iff their distance is 1.

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Let $\mathcal{M}$ be s.t. $G_{\mathcal{M}}$ is a cycle of length 5.

**Open Question.** Is $\mathcal{M}$ isometrically embeddable in $(\mathcal{D}, H)$?

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A *perfect tree* is a $T : 2^{<\omega} \to 2^{<\omega}$ s.t. $T(\sigma 0)$ and $T(\sigma 1)$ are incompatible extensions of $T(\sigma)$.

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A related question: Is there a perfect tree of pairwise relatively 1-random sets?

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How do we obtain ideals that are jump-cofinal but not cofinal?

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Let $(A)^n$ be the set of ordered $n$-tuples of distinct elements of $A$.

**Thm (Mycielski).** Let $(\mathcal{R}_i \subseteq (2^\omega)^{n_i})_{i\in\omega}$ be s.t. each $\mathcal{R}_i$ has measure 1. There is a nonempty perfect $\mathcal{C} \subseteq 2^\omega$ s.t. $(\mathcal{C})^{n_i} \subseteq \mathcal{R}_i$ for all $i$.

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Can we do the same using 1-randomness?

Let $(A)^n$ be the set of ordered $n$-tuples of distinct elements of $A$.

**Thm (Mycielski).** Let $(\mathcal{R}_i \subseteq (2^\omega)^{n_i})_{i\in\omega}$ be s.t. each $\mathcal{R}_i$ has measure 1. There is a nonempty perfect $\mathcal{C} \subseteq 2^\omega$ s.t. $(\mathcal{C})^{n_i} \subseteq \mathcal{R}_i$ for all $i$.

Miller and Yu gave a direct construction of a perfect tree such that every join of finitely many distinct paths is 1-random, and showed how its relativized version yields a proof of Mycielski's Theorem.

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**Thm.** If a degree has a strong minimal cover then it does not compute any perfect tree all of whose paths are 1-random.

**Thm (Barmpalias and Lewis).** Every 2-random degree has a strong minimal cover.

**Cor.** No 2-random computes a perfect tree all of whose paths are 1-random.