The Cantor-Bendixson theorem in the Weihrauch lattice

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The project

In a 2015 Dagstuhl seminar I asked "What do the Weihrauch hierarchies look like once we go to very high levels of reverse mathematics strength?"

In other words, I proposed to study the multi-valued functions arising from theorems which lie around ATR_0 and $\mathbf{\Pi}^1_1$-CA_0.

People who have worked to this project, mainly at the level of ATR_0, so far include Takayuki Kihara, Arno Pauly, Jun Le Goh, Jeff Hirst, Paul-Elliot Anglès d'Auriac, and Manlio Valenti.

We are now moving to the level of $\mathbf{\Pi}^1_1$-CA_0.

Outline

Represented spaces

A representation σ_X of a set X is a surjective partial function $\sigma_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$.

The pair (X, σ_X) is a represented space.

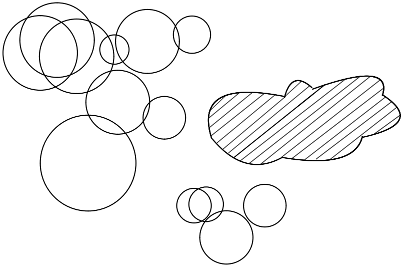
If $x \in X$ a σ_X-name for x is any $p \in \mathbb{N}^{\mathbb{N}}$ such that $\sigma_X(p) = x$.

Representations are analogous to the codings used in reverse mathematics to speak about various mathematical objects in subsystems of second order arithmetic.

The negative representation of closed sets

Let (X, α, d) be a computable metric space.

In the negative representation of the set $\mathcal{A}^-(X)$ of closed subsets of X a name for the closed set C is a sequence of open balls with center in D and rational radius whose union is $X \setminus C$.



When $X = \mathbb{N}^\mathbb{N}$ or $X = 2^\mathbb{N}$ the negative representation is computably equivalent to the representation of C by a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ such that $[T] = C$.

Realizers

If (X, σ_X) and (Y, σ_Y) are represented spaces and $f :\subseteq X \rightrightarrows Y$ a realizer for f is a function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $\sigma_Y(F(p)) \in f(\sigma_X(p))$ whenever $f(\sigma_X(p))$ is defined, i.e. whenever p is a name of some $x \in \operatorname{dom}(f)$.

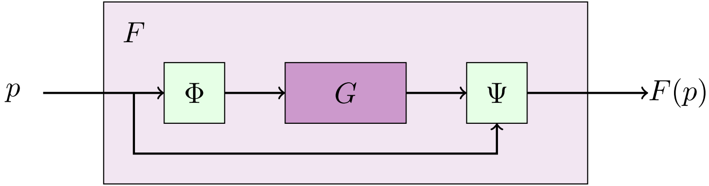
$$\begin{array}{ccc} p \in \mathbb{N}^\mathbb{N} & \xrightarrow{F} & F(p) \in \mathbb{N}^\mathbb{N} \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ x \in X & \underset{f}{\rightrightarrows} & y \in f(x) \end{array}$$

Notice that different names of the same $x \in \operatorname{dom}(f)$ might be mapped by F to names of different elements of $f(x)$.

f is computable if it has a computable realizer.

Weihrauch reducibility

Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be partial multi-valued functions between represented spaces. $f \leq_W g$ means that the problem of computing f can be computably and uniformly solved by using in each instance a single computation of g.



If G is a realizer for g then F is a realizer for f.

1. $\Phi :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a computable function that modifies (a name for) the input of f to feed it to g;
2. $\Psi :\subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a computable function that, using also (the name for) the original input, transforms (the name of) any output of g into (a name for) a correct output of f.

Arithmetic Weihrauch reducibility

Arithmetic Weihrauch reducibility $\leq^a_{\mathrm{W}}$ is obtained from Weihrauch reducibility by relaxing the condition on Ψ and Φ and requiring them to be arithmetic rather than computable.

It is immediate that $f \leq_{\mathrm{W}} g$ implies $f \leq^a_{\mathrm{W}} g$.

Arithmetic Weihrauch reducibility was introduced by Kihara-Anglès D'Auriac and independently by Goh.

The Weihrauch lattice

$\leq_{\mathrm{W}}$ is reflexive and transitive and induces the equivalence relation $\equiv_{\mathrm{W}}$. The $\equiv_{\mathrm{W}}$-equivalence classes are called Weihrauch degrees.

The partial order on the sets of Weihrauch degrees is a distributive bounded lattice with several natural and useful algebraic operations: the Weihrauch lattice.

Products

The parallel product of $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ is $f \times g :\subseteq X \times Z \rightrightarrows Y \times W$ defined by

$$(f \times g)(x, z) = f(x) \times g(z).$$

The compositional product $f \star g$ satisfies

$$f \star g \equiv_{\mathrm{W}} \max_{\leq_{\mathrm{W}}} \{f_1 \circ g_1 \mid f_1 \leq_{\mathrm{W}} f \wedge g_1 \leq_{\mathrm{W}} g\}$$

and thus is the hardest problem that can be realized using first g, then something computable, and finally f.

Parallelization

If $f :\subseteq X \rightrightarrows Y$ is a multi-valued function, the (infinite) parallelization of f is the multi-valued function $\widehat{f} : X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}$ with $\mathrm{dom}(\widehat{f}) = \mathrm{dom}(f)^{\mathbb{N}}$ defined by $f((x_n)_{n\in\mathbb{N}}) = \prod_{n\in\mathbb{N}} f(x_n)$.

$\widehat{f}$ computes f countably many times in parallel.

f is parallelizable if $\widehat{f} \equiv_{\mathrm{W}} f$.

The finite parallelization of f is the multi-valued function $f^* : X^* \rightrightarrows Y^*$ where $X^* = \bigcup_{i\in\mathbb{N}}(\{i\} \times X^i)$ with $\mathrm{dom}(f^*) = \mathrm{dom}(f)^*$ defined by $f^*(i, (x_j)_{j<i}) = \{i\} \times \prod_{j<i} f(x_j)$.

The unbounded finite parallelization of f is a multi-valued function f^{*u} (recently introduced by Soldà and Valenti) which behaves as f^* but does not bound a priori the number of instances of f that will be used.

Some examples

- The limited principle of omniscience is the function $\mathsf{LPO} : \mathbb{N}^{\mathbb{N}} \to 2$ such that $\mathsf{LPO}(p) = 0$ iff $\forall i\, p(i) = 0$.
- $\lim : \subseteq (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ maps a convergent sequence in Baire space to its limit.

$\lim$ is parallelizable, while LPO is not (and in fact $\widehat{\mathsf{LPO}} \equiv_{\mathrm{W}} \lim$).

Choice functions

Let X be a computable metric space and recall that $\mathcal{A}^-(X)$ is the space of its closed subsets represented by negative information.

$\mathsf{C}_X :\subseteq \mathcal{A}^-(X) \rightrightarrows X$ is the choice function for X: it picks from a nonempty closed set in X one of its elements.

$\mathsf{UC}_X :\subseteq \mathcal{A}^-(X) \to X$ is the unique choice function for X: it picks from a singleton (represented as a closed set) in X its unique element (in other words, UC_X is the restriction of C_X to singletons).

$\mathsf{TC}_X : \mathcal{A}^-(X) \rightrightarrows X$ is the total continuation of the choice function for X: it extends C_X by setting $\mathsf{TC}_X(\emptyset) = X$.

In general we have $\mathsf{UC}_X \leq_{\mathrm{W}} \mathsf{C}_X \leq_{\mathrm{W}} \mathsf{TC}_X$ and, for example, $\mathsf{UC}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}$, $\mathsf{UC}_{2^{\mathbb{N}}} <_{\mathrm{W}} \mathsf{C}_{2^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{TC}_{2^{\mathbb{N}}}$ and $\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}} <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} <_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^{\mathbb{N}}}$.

The Weihrauch lattice and reverse mathematics

We can locate theorems in the Weihrauch lattice by looking at the multi-valued functions they naturally translate into.

In most cases the Weihrauch lattice refines the classification provided by reverse mathematics: statements equivalent over RCA_0 may give rise to functions with different Weihrauch degrees.

Weihrauch reducibility is finer because requires both uniformity and use of a single instance of the harder problem.

- computable functions correspond to RCA_0;
- $\mathsf{C}_{2^{\mathbb{N}}}$ corresponds to WKL_0;
- $\lim$ and its iterations correspond to ACA_0;
- the interval of the Weihrauch lattice between $\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$ and $\mathsf{TC}^*_{\mathbb{N}^{\mathbb{N}}}$ corresponds to ATR_0.

Arithmetical Transfinite Recursion in the Weihrauch lattice

ATR is the function producing, for a well-order X, a jump hierarchy along X.

Theorem (Kihara-M-Pauly 2020)

$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{ATR}$.

ATR_2 is the function producing, for a linear order X, either a jump hierarchy along X or a descending sequence in X.

Theorem (Goh 2019)

$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} \mathsf{ATR}_2 <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ *and the inequalities are strict even with respect to arithmetic reducibility.*

The function corresponding to $\mathbf{\Pi}^1_1$-CA_0

Tr is the set of subtrees of $\mathbb{N}^{<\mathbb{N}}$.

- $\chi_{\Pi^1_1} : \mathrm{Tr} \to 2$ such that $\chi_{\Pi^1_1}(T) = 1$ iff T is well-founded.
- $\mathbf{\Pi}^1_1$-$\mathsf{CA} = \widehat{\chi_{\Pi^1_1}} : \mathrm{Tr}^{\mathbb{N}} \to 2^{\mathbb{N}}$ maps $(T_n)_{n\in\mathbb{N}}$ to the characteristic function of $\{ n \in \mathbb{N} \mid T_n \text{ is well-founded} \}$.

$\mathbf{\Pi}^1_1$-CA is the natural candidate to correspond to $\mathbf{\Pi}^1_1$-CA_0.

The perfect tree theorem

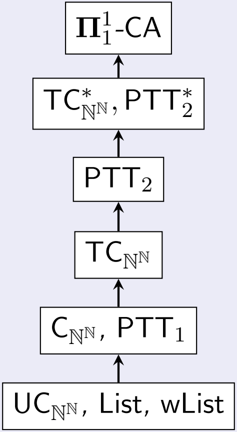
The Perfect Tree Theorem asserts that if $T \in \mathrm{Tr}$, then either $[T]$ is countable or T has a perfect subtree.

In reverse mathematics it is equivalent to ATR_0.

- $\mathsf{PTT}_1 : \subseteq \mathrm{Tr} \rightrightarrows \mathrm{Tr}$ maps a tree with uncountably many paths to the set of its perfect subtrees.
- $\mathsf{List} : \subseteq \mathrm{Tr} \rightrightarrows (\mathbb{N}^\mathbb{N})^\mathbb{N} \times \mathbb{N}$ maps a tree with no perfect subtree to a list of its paths and their number.
- $\mathsf{wList} : \subseteq \mathrm{Tr} \rightrightarrows (\mathbb{N}^\mathbb{N})^\mathbb{N}$ maps a tree with no perfect subtree to a list of its paths.
- $\mathsf{PTT}_2 : \subseteq \mathrm{Tr} \rightrightarrows \mathrm{Tr} \times (\mathbb{N}^\mathbb{N})^\mathbb{N}$ maps a tree to a pair $(T', (p_n))$ such that either T' is a perfect subtree of T or (p_n) lists $[T]$.

The perfect tree theorem in the Weihrauch lattice

Theorem (Kihara-M-Pauly 2020)

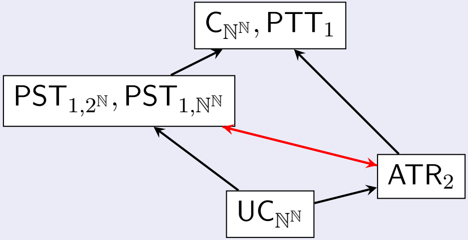


The perfect set theorem

The perfect set theorem deals with closed sets in Polish spaces.
In reverse mathematics it is equivalent to ATR_0.

For X a computable Polish space let $\mathsf{PST}_{1,X} : \mathcal{A}^-(X) \rightrightarrows \mathcal{A}^-(X)$ be the function mapping an uncountable closed set C to a perfect closed subset of C.

Theorem (Cipriani-M-Valenti)



Moreover $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \lim \star \mathsf{PST}_{1,\mathbb{N}^\mathbb{N}}$ *so that* $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}}^{a} \mathsf{PST}_{1,\mathbb{N}^\mathbb{N}}$.

Perfect kernels of trees

The Perfect Kernel Theorem asserts that if $T \in \mathrm{Tr}$, then T has a largest (possibly empty) perfect subtree, called the perfect kernel of T.

In reverse mathematics it is equivalent to $\mathbf{\Pi}^1_1$-CA_0.

Let $\mathsf{PK}_{\mathrm{Tr}} : \mathrm{Tr} \to \mathrm{Tr}$ be the function that maps a tree T to its perfect kernel.

Theorem (Hirst 2020)

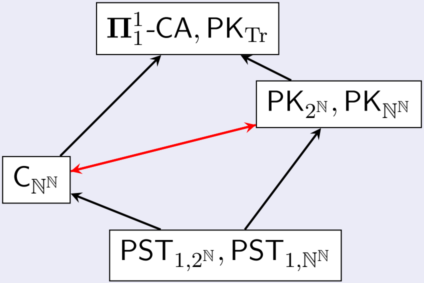
$\mathbf{\Pi}^1_1$-$\mathsf{CA} \equiv_{\mathrm{W}} \mathsf{PK}_{\mathrm{Tr}}$.

Perfect kernels of closed sets

The perfect kernel theorem extends to closed sets in Polish spaces.

For X a computable Polish space let $\mathsf{PK}_X : \mathcal{A}^-(X) \to \mathcal{A}^-(X)$ be the function mapping a closed set C to its perfect kernel, i.e. the largest perfect closed subset of C.

Theorem (Cipriani-M-Valenti)



Moreover $\mathbf{\Pi}^1_1$-CA $\leq_{\mathrm{W}} \lim \star \mathsf{PK}_{\mathbb{N}^\mathbb{N}}$ *so that* $\mathbf{\Pi}^1_1$-CA $\equiv^a_{\mathrm{W}} \mathsf{PK}_{\mathbb{N}^\mathbb{N}}$.

The Cantor-Bendixson Theorem for trees

The Cantor-Bendixson Theorem for trees asserts that if $T \in \mathrm{Tr}$, then T has a (possibly empty) perfect subtree T' such that $[T] \setminus [T']$ is countable. T' is in fact the perfect kernel of T and $[T] \setminus [T']$ is called the scattered part of T.

In reverse mathematics it is equivalent to $\mathbf{\Pi}^1_1$-CA_0.

$\mathsf{CB}_{\mathrm{Tr}} : \mathrm{Tr} \rightrightarrows \mathrm{Tr} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \times \mathbb{N}$ maps a tree T to the perfect kernel of T, a list of the scattered part of T and its size.

$\mathsf{wCB}_{\mathrm{Tr}} : \mathrm{Tr} \rightrightarrows \mathrm{Tr} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ maps a tree T to the perfect kernel of T and a list of the scattered part of T.

Theorem (Cipriani-M-Valenti)

$\mathbf{\Pi}^1_1$-$\mathsf{CA} \equiv_{\mathrm{W}} \mathsf{wCB}_{\mathrm{Tr}} \equiv_{\mathrm{W}} \mathsf{CB}_{\mathrm{Tr}}$.

The Cantor-Bendixson Theorem for closed sets

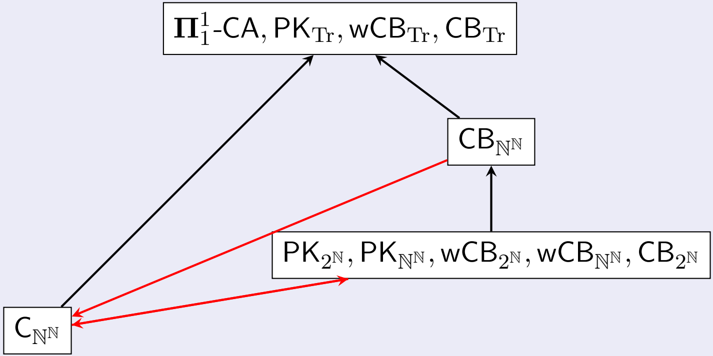
The Cantor-Bendixson Theorem also extends to closed sets in Polish spaces.

For X a computable Polish space
$\mathsf{CB}_X : \mathcal{A}^-(X) \rightrightarrows \mathcal{A}^-(X) \times X^{\mathbb{N}} \times \mathbb{N}$ maps a closed set C to the perfect kernel of C, a list of the scattered part of C and its size.

$\mathsf{wCB}_X : \mathrm{Tr} \rightrightarrows \mathcal{A}^-(X) \rightrightarrows \mathcal{A}^-(X) \times X^{\mathbb{N}}$ maps a closed set C to the perfect kernel of C and a list of the scattered part of C.

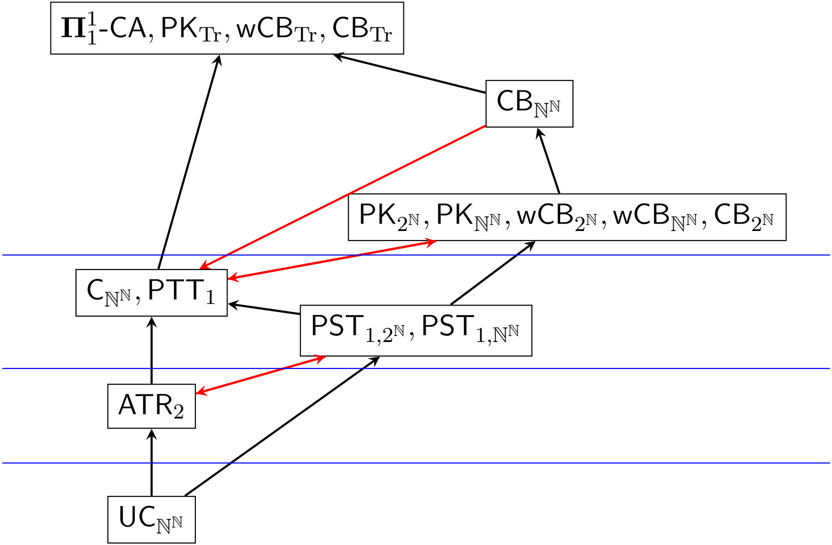
The Cantor-Bendixson Theorem for closed sets in the Weihrauch lattice

Theorem (Cipriani-M-Valenti)



We do not know whether $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_{\mathrm{W}} \mathsf{CB}_{\mathbb{N}^\mathbb{N}}$.

Summing up


$\mathbf{\Pi}^1_1$-CA, $\mathsf{PK}_{\mathrm{Tr}}$, $\mathsf{wCB}_{\mathrm{Tr}}$, $\mathsf{CB}_{\mathrm{Tr}}$
$\mathsf{CB}_{\mathbb{N}^\mathbb{N}}$
$\mathsf{PK}_{2^\mathbb{N}}$, $\mathsf{PK}_{\mathbb{N}^\mathbb{N}}$, $\mathsf{wCB}_{2^\mathbb{N}}$, $\mathsf{wCB}_{\mathbb{N}^\mathbb{N}}$, $\mathsf{CB}_{2^\mathbb{N}}$
$\mathsf{C}_{\mathbb{N}^\mathbb{N}}$, PTT_1
$\mathsf{PST}_{1,2^\mathbb{N}}$, $\mathsf{PST}_{1,\mathbb{N}^\mathbb{N}}$
ATR_2
$\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$

The first order part

Definition (Dzhafarov-Solomon-Yokoyama)

The first order part of $f :\subseteq X \rightrightarrows Y$ is a specific function ${}^1f :\subseteq \mathbb{N}^\mathbb{N} \times X \rightrightarrows \mathbb{N}$ such that its Weihrauch degree is $\max\{\, g \mid g \leq_\mathrm{W} f \wedge \text{ the codomain of } g \text{ is } \mathbb{N} \,\}$

Theorem (Soldà-Valenti)

If $f = \widehat{h}$ then ${}^1f \equiv_\mathrm{W} h^{*u}$.
If h is total, pointed and with codomain $\mathbb{N}$, then $h^{*u} \equiv_\mathrm{W} h^\diamond$.

Since $\mathbf{\Pi}^1_1$-CA $\equiv_\mathrm{W} \widehat{\chi_{\Pi^1_1}}$, we have ${}^1\mathbf{\Pi}^1_1$-CA $= \chi^{*u}_{\Pi^1_1} = \chi^\diamond_{\Pi^1_1}$.

On the other hand, we show that ${}^1\mathsf{PK}_{\mathbb{N}^\mathbb{N}} \equiv_\mathrm{W} \mathbf{\Pi}^1_1$-$\mathsf{C}_\mathbb{N} <_\mathrm{W} \chi^{*u}_{\Pi^1_1}$.

This yields $\mathbf{\Pi}^1_1$-CA $\not\leq_\mathrm{W} \mathsf{PK}_{\mathbb{N}^\mathbb{N}}$.

The deterministic part

Definition (Goh-Pauly-Valenti)

The deterministic part of $f :\subseteq X \rightrightarrows Y$ is a specific function $\mathrm{Det}(f) :\subseteq \mathbb{N}^{\mathbb{N}} \times X \to \mathbb{N}^{\mathbb{N}}$ such that its Weihrauch degree is $\max\{\, g \mid g \leq_{\mathrm{W}} f \wedge \text{ the codomain of } g \text{ is } \mathbb{N}^{\mathbb{N}} \wedge g \text{ is single-valued} \,\}$

We use the deterministic part to show $\mathbf{\Pi}^1_1$-CA $\not\leq_{\mathrm{W}}$ $\mathsf{CB}_{\mathbb{N}^{\mathbb{N}}}$ by proving that the solutions to $\mathrm{Det}(\mathsf{CB}_{\mathbb{N}^{\mathbb{N}}})$ are hyperarithmetic in the input.

The completion

In the proof of $\mathsf{wCB}_{2^{\mathbb{N}}} \leq_{\mathrm{W}} \mathsf{PK}_{2^{\mathbb{N}}}$ we use the completion of a multi-valued function (originally due to Dzhafarov and Brattka-Gherardi).

Direction for further research

- Does $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_{\mathrm{W}} \mathsf{CB}_{\mathbb{N}^\mathbb{N}}$?
- Extend the research to computable Polish spaces other than $\mathbb{N}^\mathbb{N}$ and $2^\mathbb{N}$. Here CB_X is probably the most interesting function.
- Move on to other theorems equivalent to $\mathbf{\Pi}^1_1$-CA_0.

The end

Thank you for your attention!