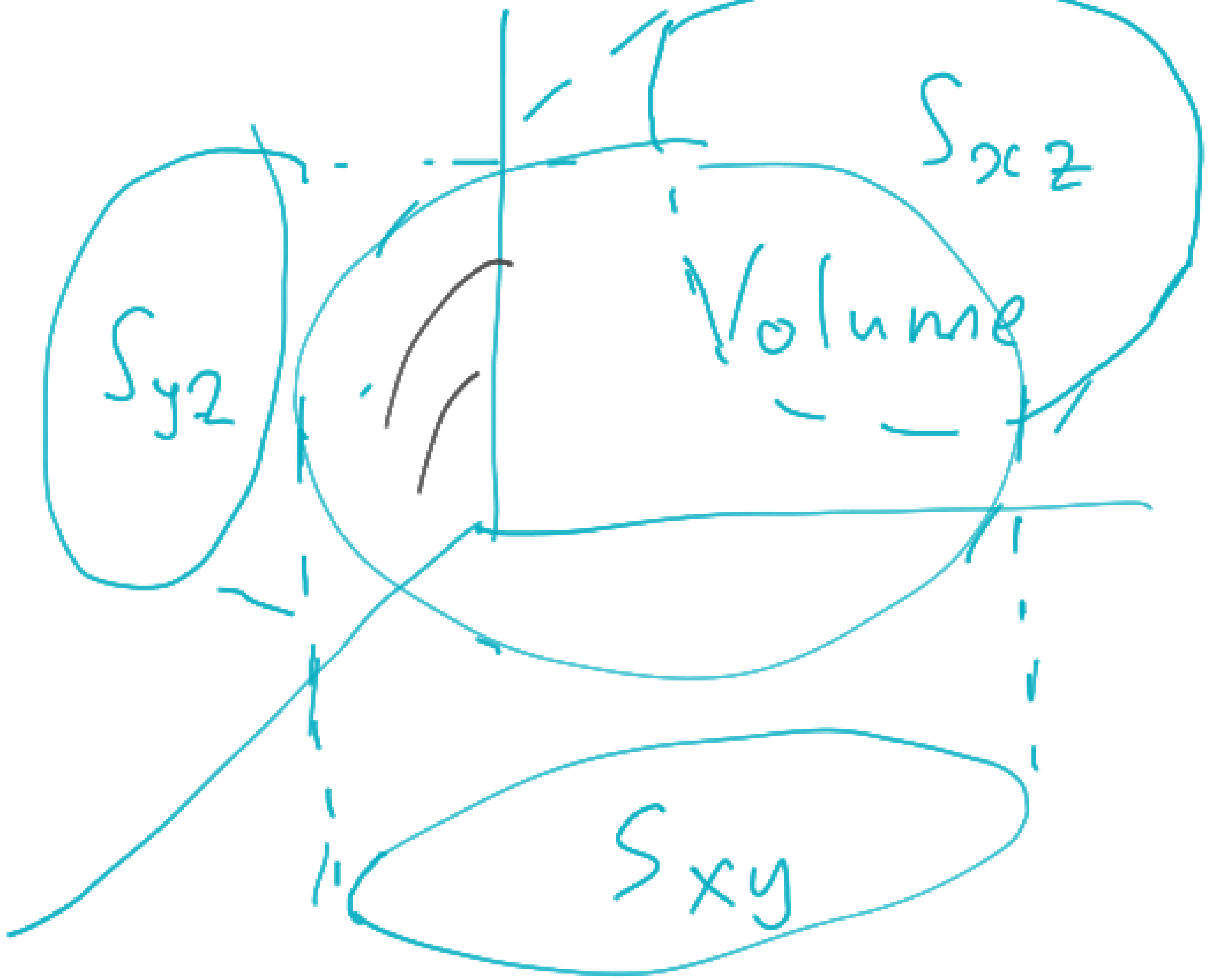
Lines and points

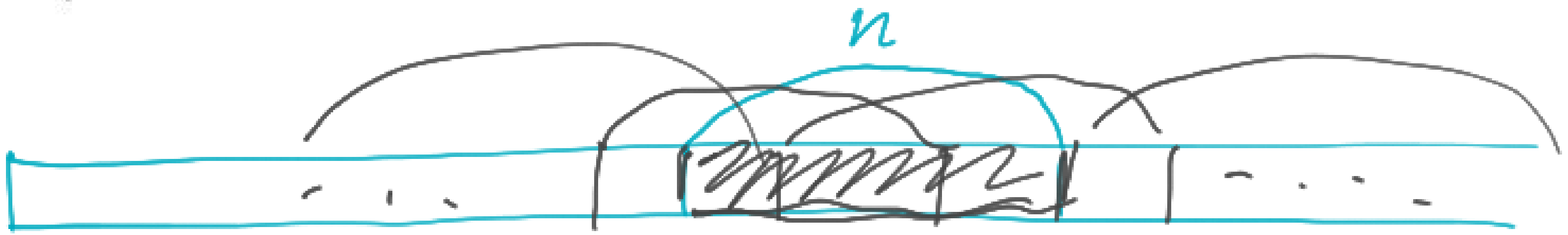
discussion with Linda Westrick, Jan Reimann, Kolmogorov seminar et al.

Thanks & Apologies

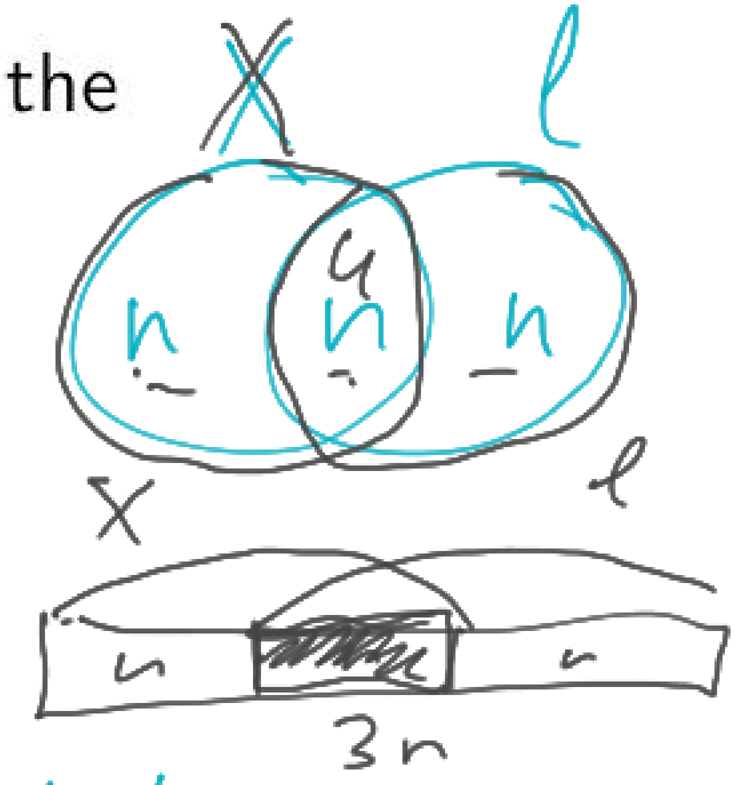
- $2\,\mathrm{C}(x, y, z) \leqslant \mathrm{C}(x, y) + \mathrm{C}(y, z) + \mathrm{C}(z, x)$
- $V^2 \leqslant S_{xy} S_{xz} S_{xz}$

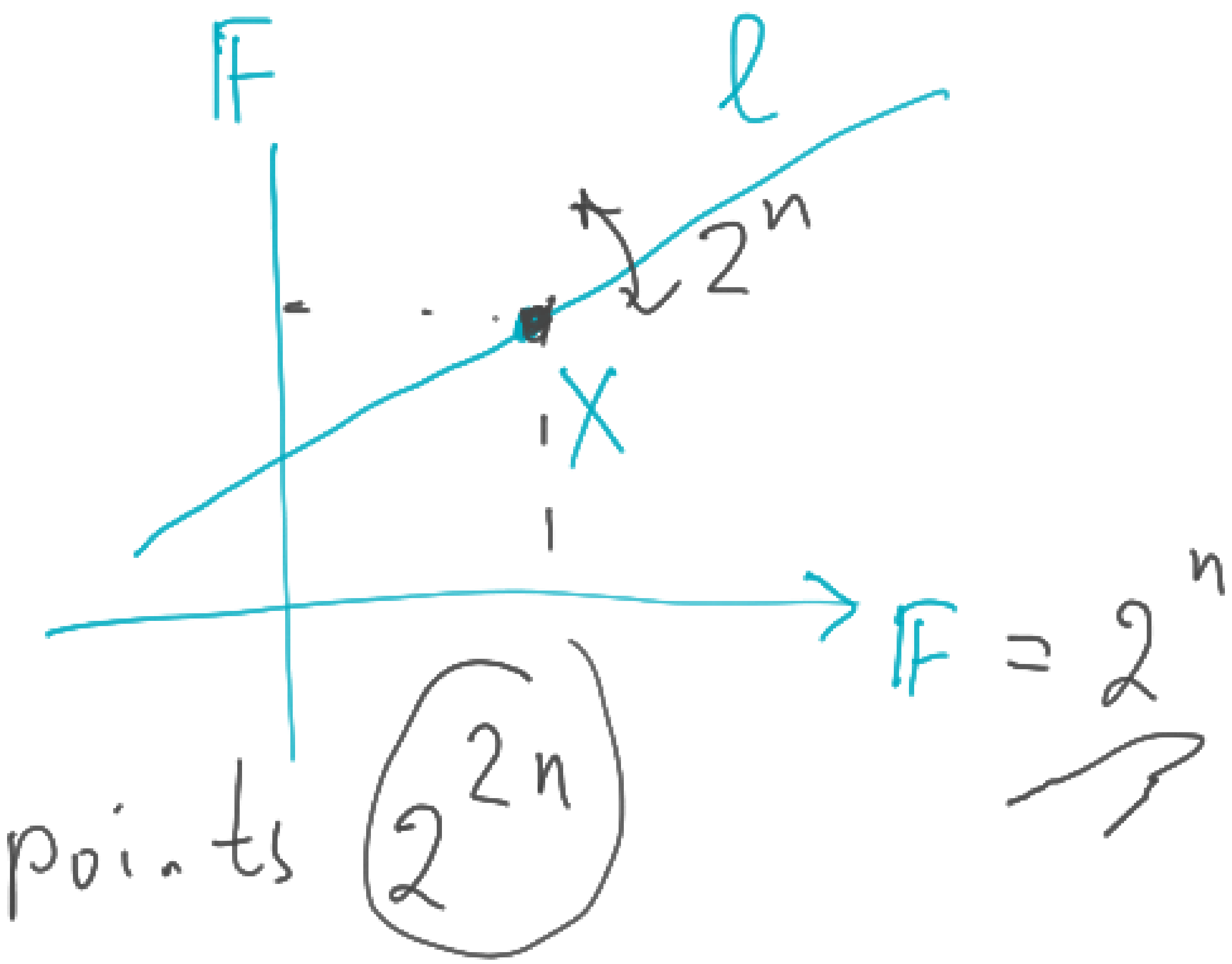


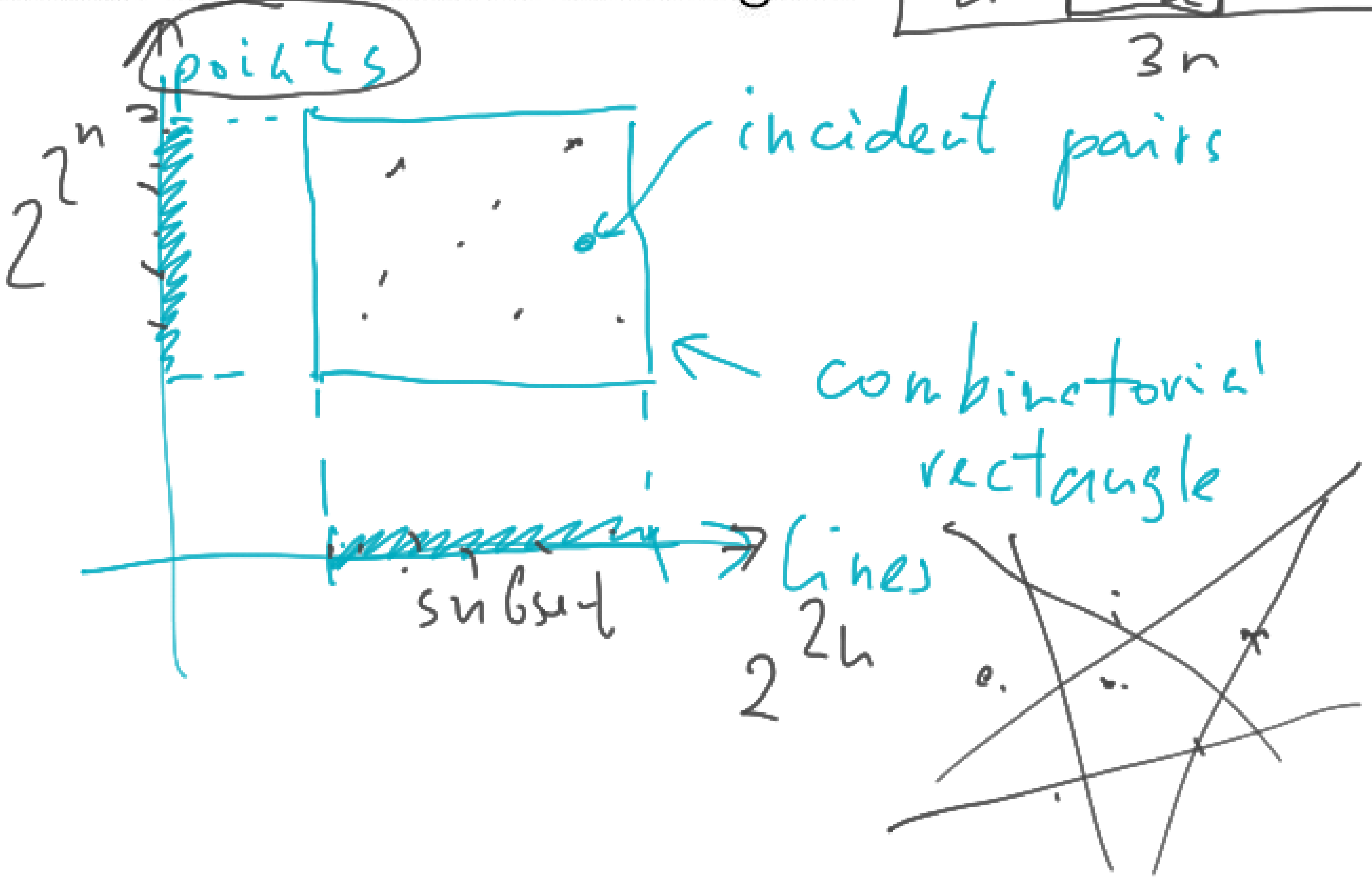
- there exist an everywhere complex sequence: any factor of length n has complexity $0.99n - O(1)$
- Lovasz local lemma: extension of union bound with partial independence



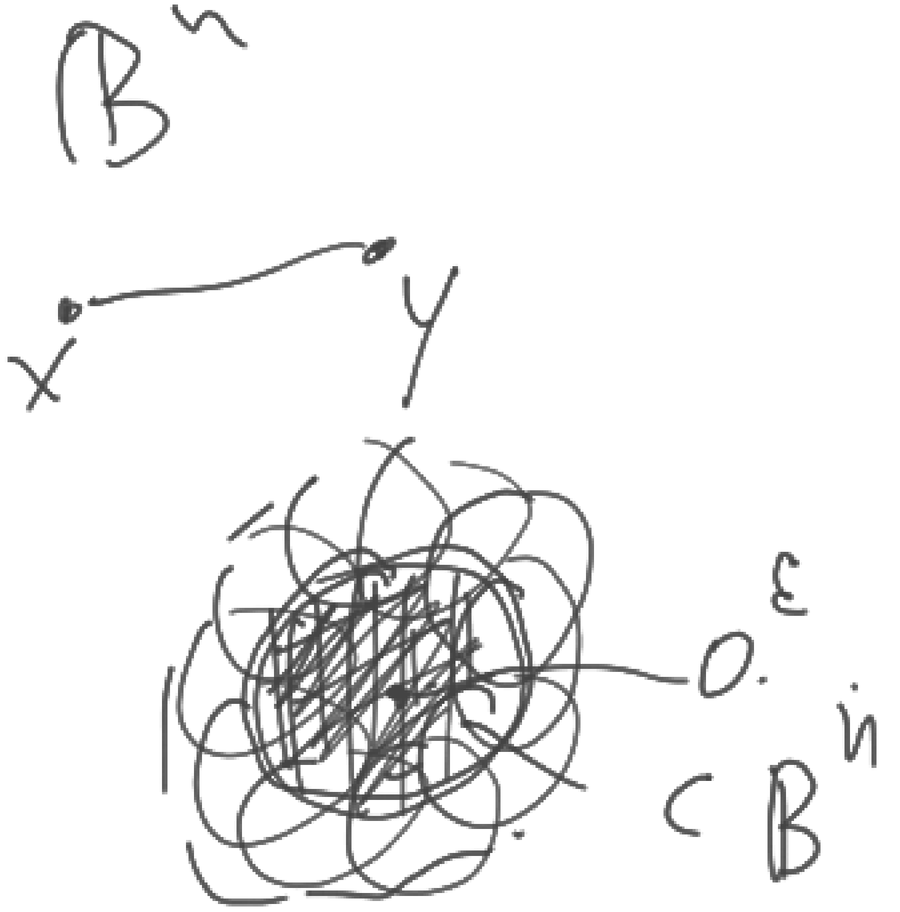
- let (X, l) be a random pair of incident point and line on the affine plane $\mathbb{F}^2$ with $\#\mathbb{F} = 2^n$
  $\mathrm{C}(X) = 2n$, $\mathrm{C}(l) = 2n$, $\mathrm{C}(X, l) = 3n$, $I(X : l) = n$
  there is no string u such that $\mathrm{C}(u) \approx n$, $\mathrm{C}(X|u) \approx n$, $\mathrm{C}(l|u) \approx n$
- incidence graph is sparse inside combinatorial rectangles

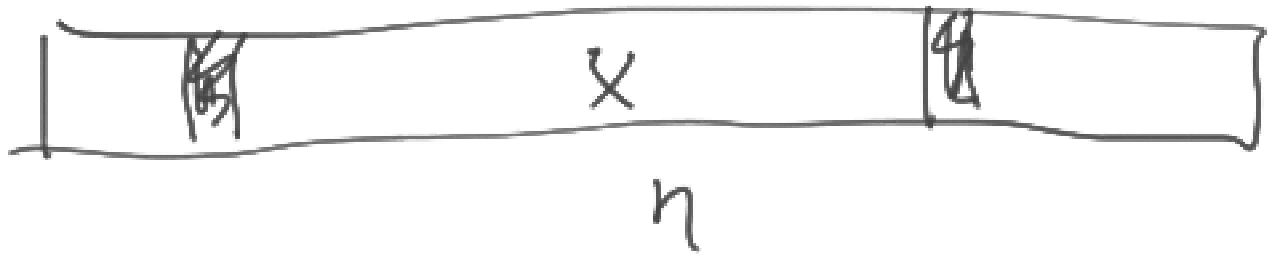






- every n-bit string of complexity αn (where $\alpha < 1$) can be made more complex by changing ε-fraction of bits [optimal bounds]
- Harper's theorem: Hamming balls have minimal ε-neighborhoods among all sets of given size



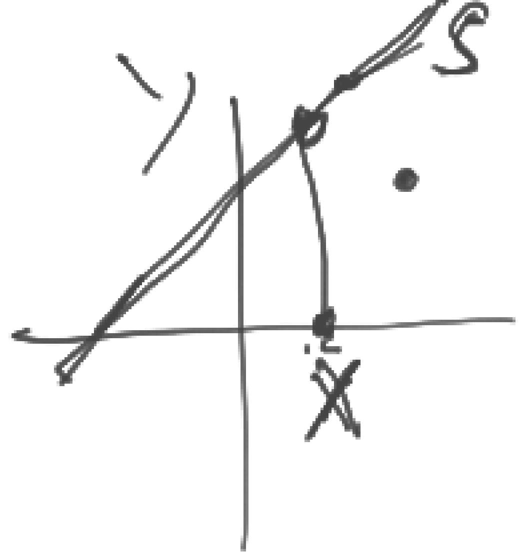


ε-fraction

$C(x) = \alpha n$

$\alpha < 1$

- For every line of Hausdorff dimension s the maximal Hausdorff dimension of its points is $\min(1+s, 2)$ [N. Lutz & D. Stull]
- What about finite planes? $1 \le s \le 2 \to 2$, $s < 1 \to 1+s$



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Proposition

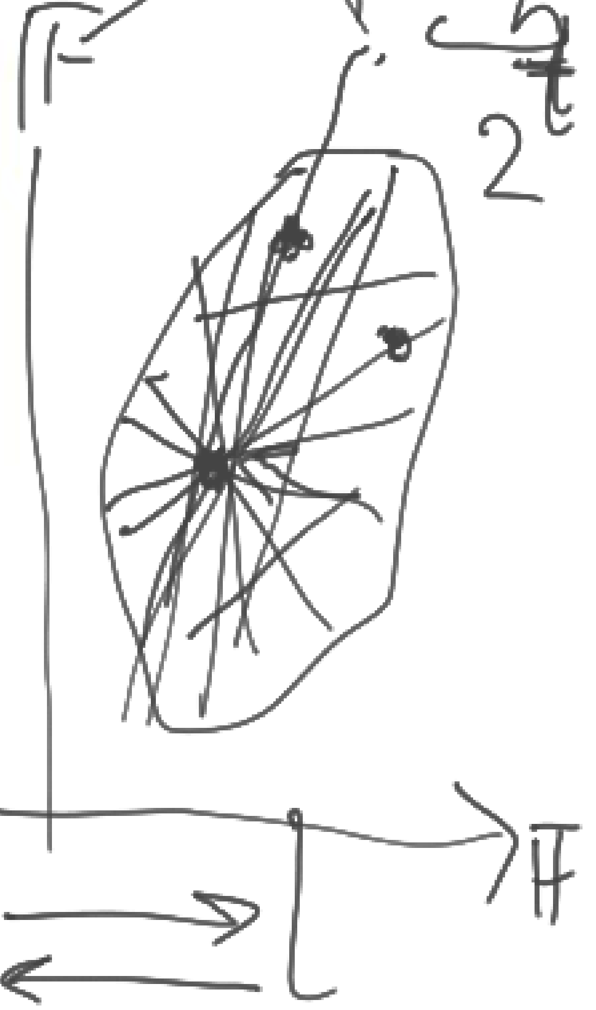
For a line in $\mathbb{F}^2$ (where $\#\mathbb{F} \approx 2^n$) of complexity s the maximal complexity of points is $s+n$ or $2n$, whichever is smaller, plus $O(\log n)$.

~~liminf~~ $\frac{C(x_1 \ldots x_n)}{n}$

Proposition

For $t < 2n$, if a set of points has less than $2^t/\operatorname{poly}(n)$ elements, then at most $2^{t-n}\operatorname{poly}(n)$ lines lie entirely in this set.

2^t



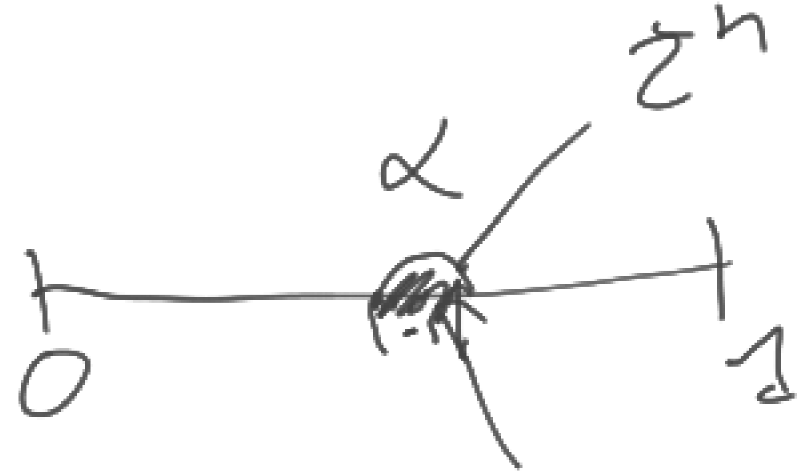
2^n

2^t

- Exclusion–inclusion formula (two lines intersect only in one point)
- Spectrum of the corresponding graph walk

$t < 2n \to 2^{t-n}$ lines

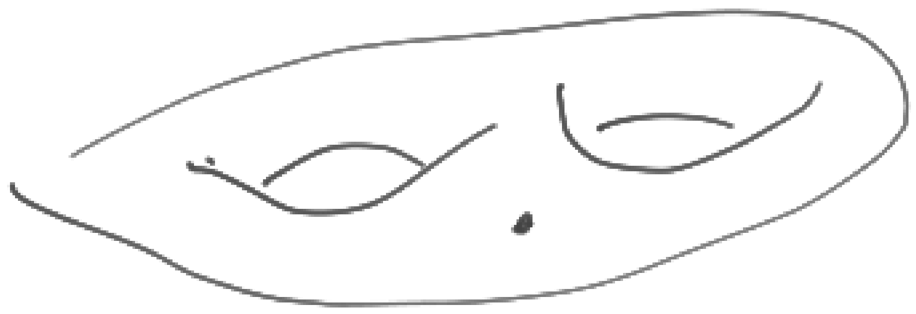
Finite continuous case

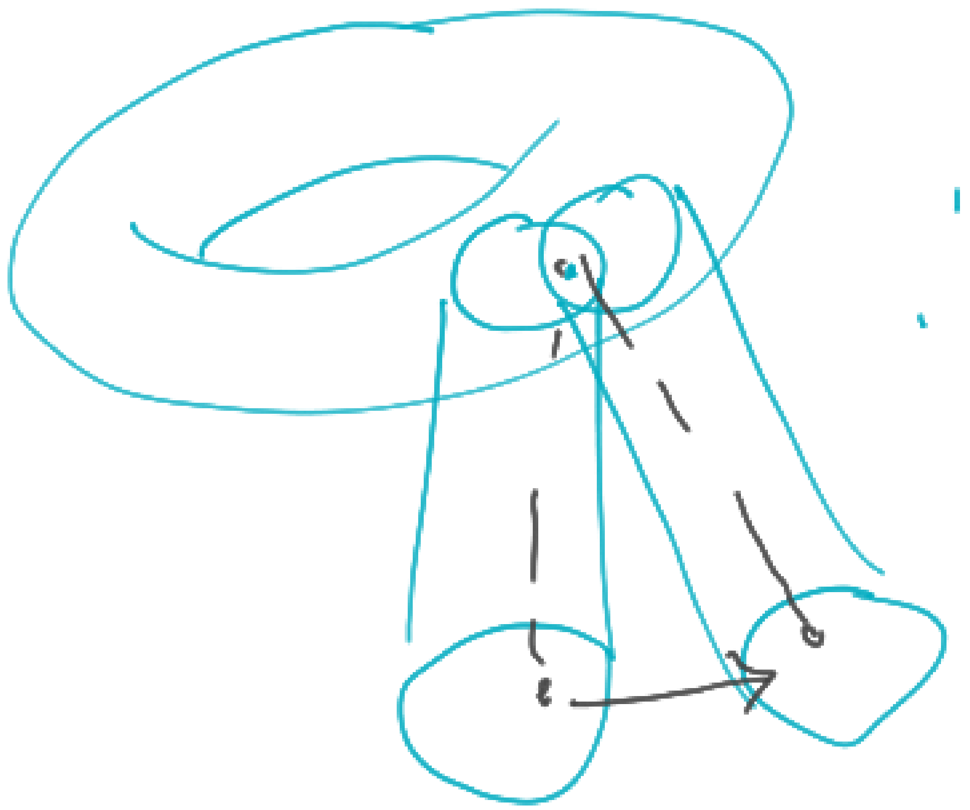


- For $\alpha \in [0,1]$: $\mathrm{C}_n(\alpha) = \mathrm{C}(\text{first } n \text{ bits of } \alpha)$
- $=$ minimal complexity of 2^{-n}-approximation

rational

- similar for points, lines, points on "reasonable" compact manifolds

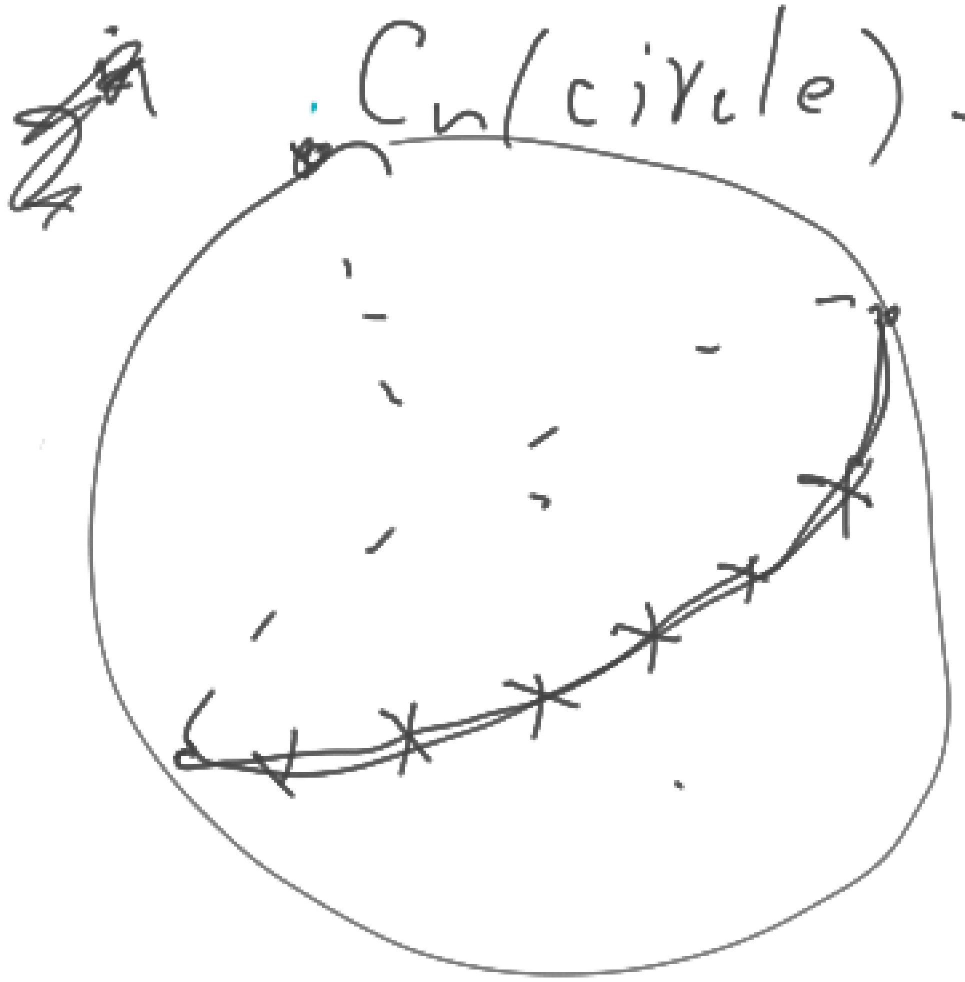




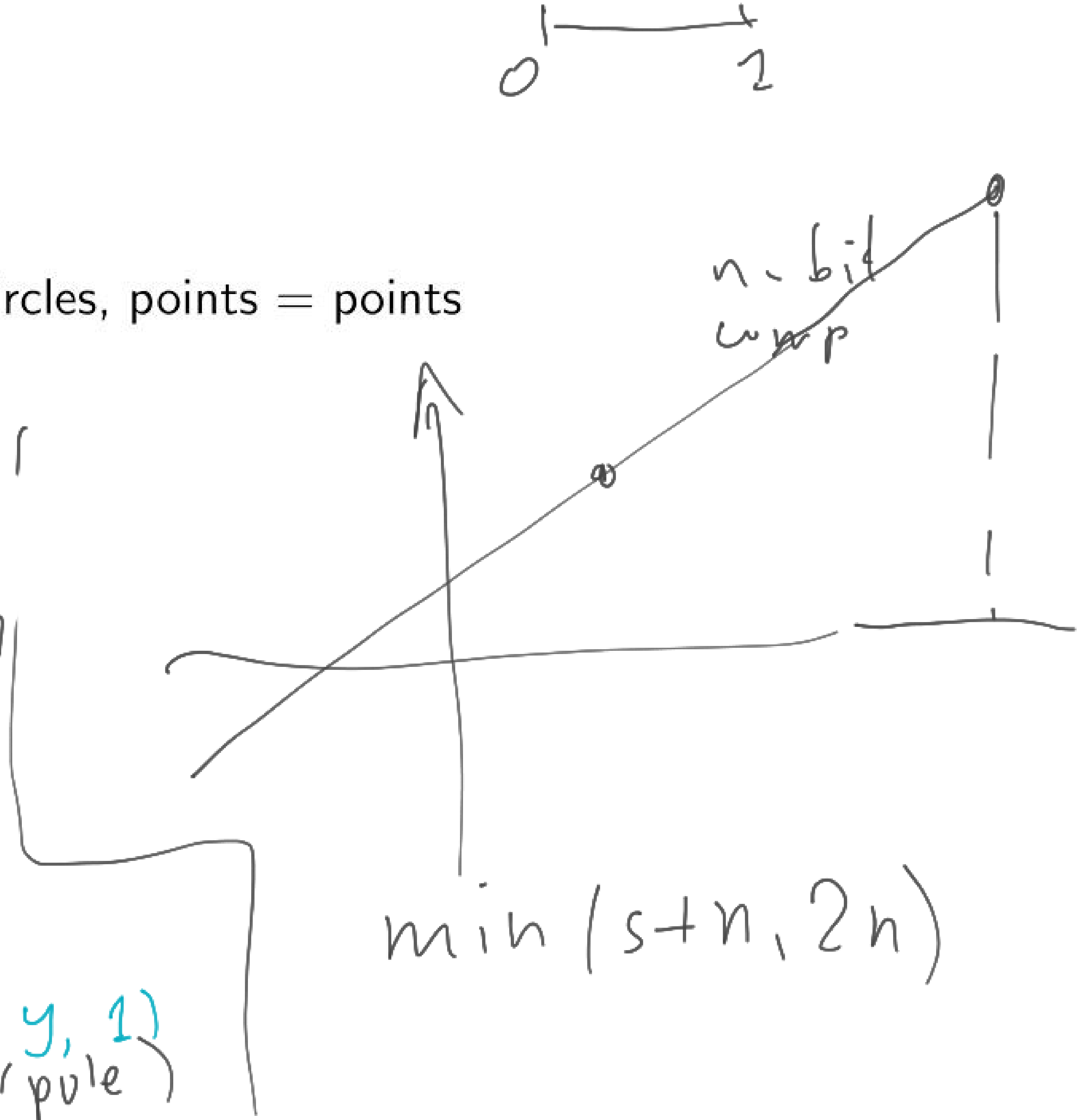
Compactification

- problem of far points
- projective plane
- 2D sphere in $\mathbb{R}^3$: lines = big circles, points = points
- $\mathrm{C}_n(\text{big circle}) := \mathrm{C}_n(\text{pole})$

$C_n(\text{circle}) \leq 2n$



$s \leq 2n$



$\min(s+n, 2n)$

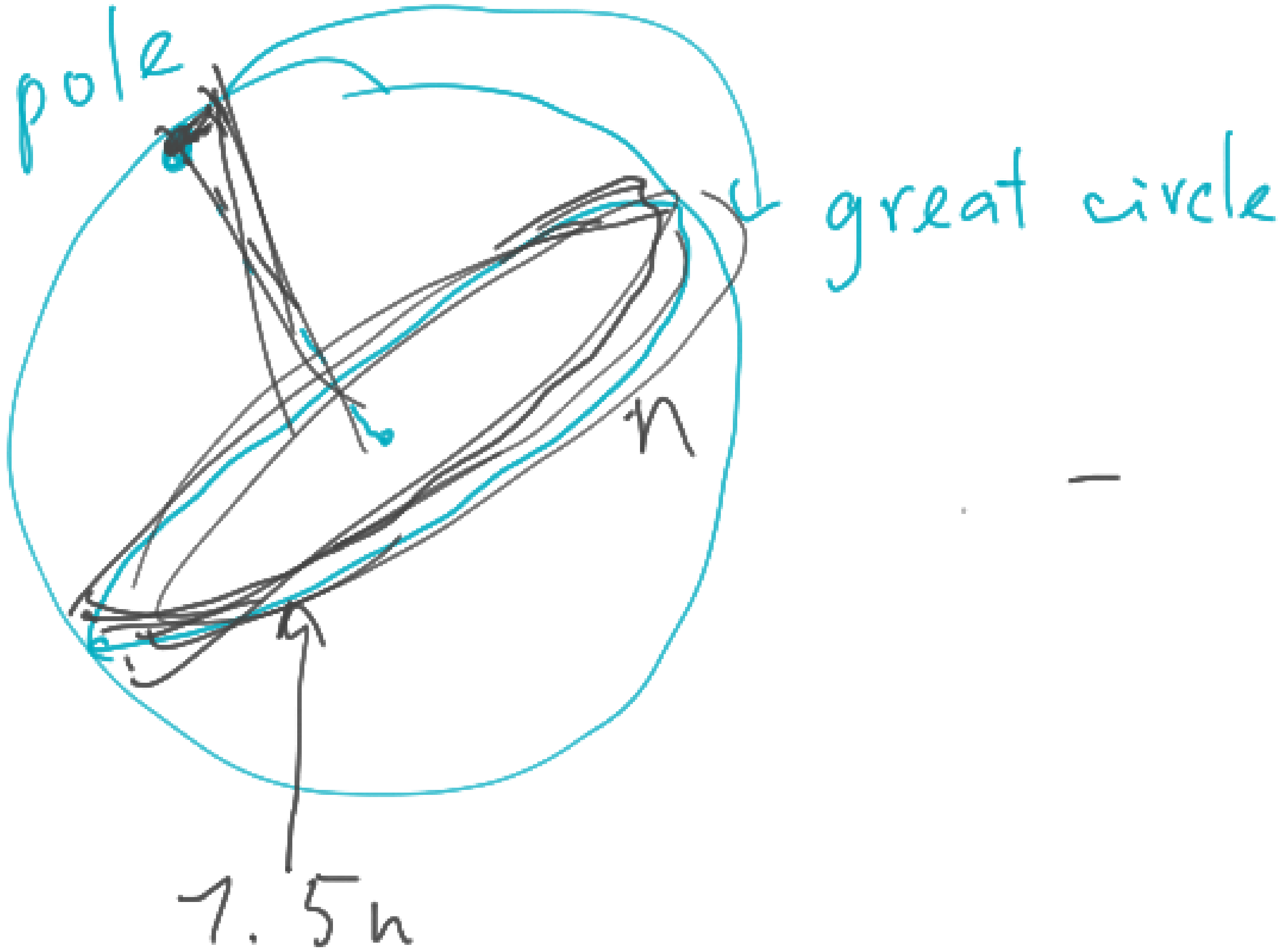
$(x, y, 1)$

n-compl (circle) = n-compl (pole)

Proposition

For a big circle of C_n-complexity s, the maximal C_n-complexity of its points is at least $n + s/2$

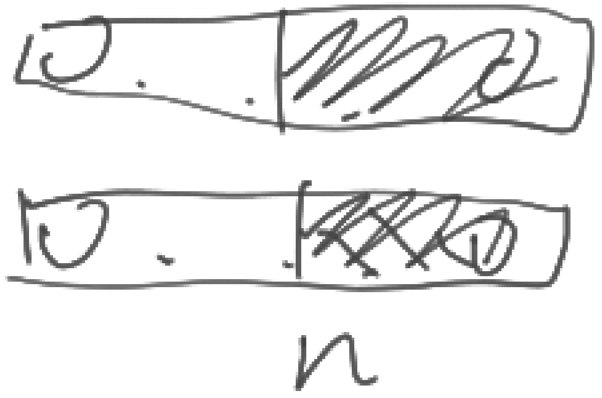
Much worse: $s/2$ in place of s. For some lines it is $n + s$, but for other the bound is tight



$y = ax + b$

$(a', b') \sim (a, b)$

$n/2$ bits kept



$(x, a'x + b')$

$(x, ax + b)$

$1.5n$

Combinatorial translation: measure of a set A vs. measure of the set of all great circles entirely in A

Proposition (Ilya Bogdanov?)

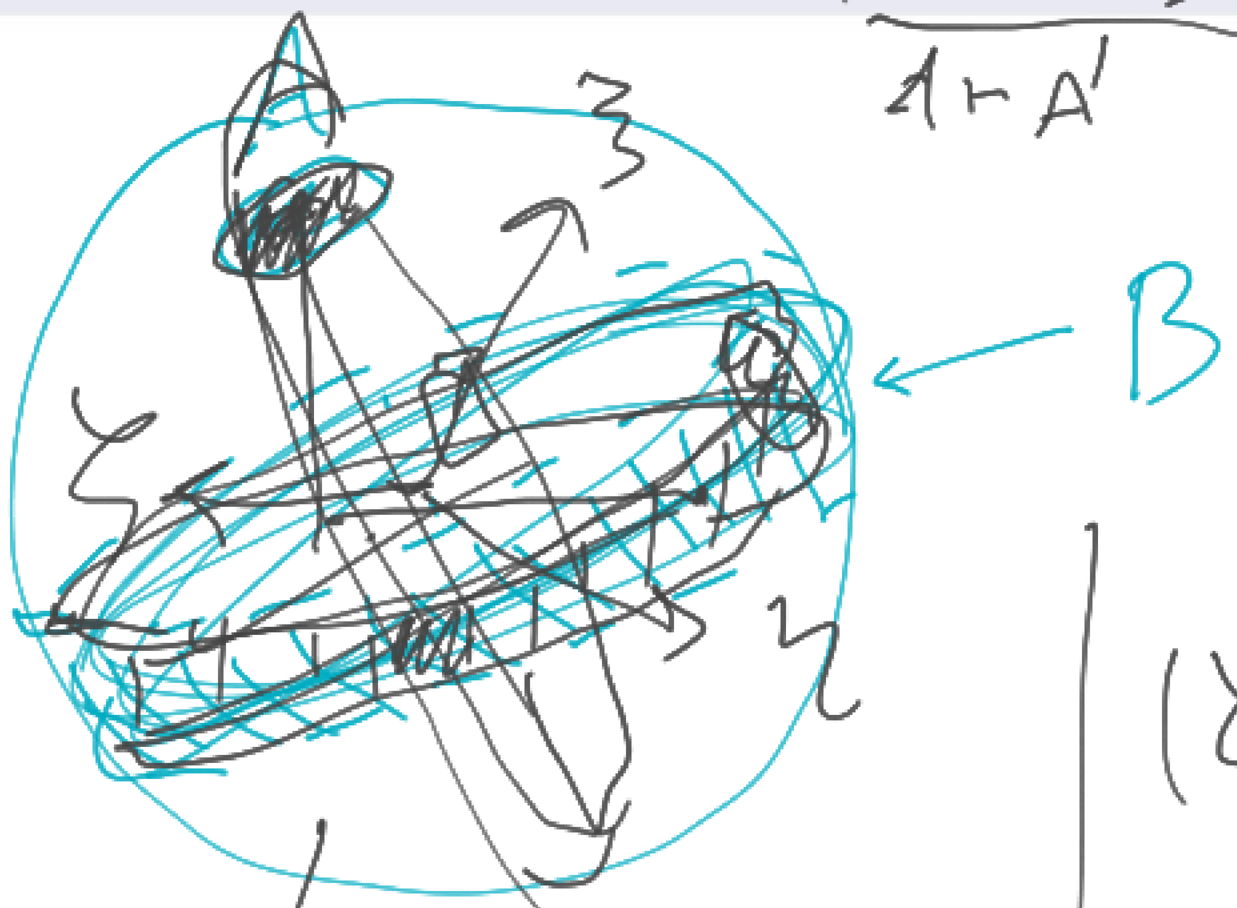
Let A and B be two measurable sets on the sphere with uniform distribution μ. Assume that for all $a \in A$, all points orthogonal to a are in B. Then

$$\mu(A) \leq \mu(B)^2$$

$1 - A'$ $\quad B \quad B'$

$\mu(B) = \mu(B')$ $\xi, \eta \in S^2$ – ind. rand.

$\zeta = \xi \wedge \eta$ – rand. varible uniformly



$\mu(S) = 1$

$(\zeta \in A) \Rightarrow (\xi, \eta \in B)$

$\mu(A) \leq \mu(B)^2$

Questions

- Why the difference with infinite (Hausdorff dimension) case? ?
- Which combinatorial result may imply the infinite case?
- What about the distribution of points complexity along the line?
- Other natural families of sets (=bipartite graphs), e.g., Hamming balls