Almost Theorems of Hyperarithmetic Analysis

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We assume a passing acquaintance with the language of second order arithmetic with two sorted structures $\mathcal{N} = (N, S, +, \times, <, 0, 1, \in)$ with $S \subseteq \mathcal{P}(N)$ and basic systems of Reverse mathematics such as RCA_0 (Δ^0_1-CA and Σ^0_1-Ind), ACA_0 (arithmetic comprehension) and ATR_0 (iterate arithmetic comprehension through well orderings). We deal only with models satisfying at least RCA_0

$\mathbb{N}$ is the standard model of arithmetic and we use $(\mathbb{N}, S)$ for $S \subseteq \mathcal{P}(\mathbb{N})$ to denote the standard models of second order arithmetic.

We also assume a passing acquaintance with the basic notions of recursion theory including Turing reducibility ($A \leq_T B \Leftrightarrow \exists e(\Phi^B_e = A)$ and the Turing jump ($X \mapsto X' = \{e | \Phi^X_e(e)\downarrow\}$) with its iterations $X \mapsto X^{(\alpha)}$ ($X^{\alpha+1} = (X^{(\alpha)})'$ and $X^{(\lambda)} = \oplus\{X^{(\alpha)} | \alpha < \lambda\}$) for ordinals α and limit ordinals λ recursive in X.

We begin our story with a class of principles/theories, the Theorems/Theories of Hyperarithmetic Analysis, THAs.

The View from Proof Theory/Reverse Mathematics

Considered from the viewpoint of Reverse Mathematics, the THAs lie above ACA_0 and indeed above each fixed α-iteration of the Turing jump for α a recursive ordinal but not above ATR_0 and may be incomparable with it.

The most commonly used THAs are probably

Σ^1_1-AC_0: For Φ arithmetical, $\forall n \exists X \Phi(n, X) \rightarrow \exists X \forall n \Phi(n, X^{[n]})$.

Δ^1_1-CA_0: Comprehension for Δ^1_1 predicates.

w-Σ^1_1-AC_0: For Φ arithmetical, $\forall n \exists ! X \Phi(n, X) \rightarrow \exists X \forall n \Phi(n, X^{[n]})$.

Studied by many beginning with Kreisel [1962], H. Friedman [1967], [1971], [1975],]; Steel [1978]...; Montalbán [2006], [2008]; ...

However, there is no axiomatic/proof theoretic characterization in the language of second order arithmetic even for ω-models.

Theorem (Van Wesep 1977)**.** If T is a THA then there is a strictly weaker $\hat{T}$ (even with more ω-models) which is also a THA.

The View from Recursion Theory:

We can the THAs recursion theoretically:

Definition: *Hyp*(X), the collection of all sets *hyperarithmetic in* X, consists of those sets recursive in some iteration $X^{(\alpha)}$ of the jump of X for α an ordinal recursive in X. These are also the sets Δ^1_1 in X.

Definition: A sentence (theory) T is a *theorem (theory) of hyperarithmetic analysis (THA)* if

1. For every $X \subseteq \mathbb{N}$, $(\mathbb{N}, HYP(X)) \vDash T$ and
2. If $(\mathbb{N}, S) \vDash T$ and $X \in S$ then $HYP(X) \subseteq S$.

So the second clause expresses strength – every standard model of T is closed under well ordered iterations of the Turing jump and so under hyperarithmetic reduction. So it **is like** ATR_0.

The first clause bounds the strength of T by saying that the sets hyperarithmetic in X form a model of T. So ATR_0 **is not** a THA.

Logical but no "Mathematical" theorems in THA

For many years there were no known nonlogical THAs from the mathematical literature (i.e. not mentioning classes of first order formulas or their syntactic complexity).

The first and, up until now the only example, was a result (INDEC) about indecomposability of linear orderings in Jullien's thesis [1969]. It was shown to be a THA by Montalbán (2006) who investigated its place among the older systems as well as several other logical ones using variations of Steel forcing. In [2008] he included Π^1_1-Separation and new forcing variations. More analysis was provided by Neeman [2009, 2011].

Montalbán, in Open Questions in Reverse Mathematics [2011], asked if there are any others.

New Mathematical THA: Rays in Graphs

Barnes, Goh and Shore [ta] (BGS) have now provided a whole family of THAs which are classical results in graph theory appearing in both articles and the standard textbook *Graph Theory* (Diestel) and some natural variations.

Definition: A *Graph* $G = (V, E)$ is a set V of vertices and a SIB (edge) relation E on V. If E is just IB, the graph is *directed*. A *ray* R in G is a subgraph $\langle V', E' \rangle$ (i.e. $V' \subseteq V$ and $E' \subseteq E$) and an isomorphism $f_{H'}$ from N with edges $(n, n+1)$ for $n \in N$ to H'. (So this gives a sequence $\langle f(n) \rangle$ of vertices with edges between $f(n)$ and $f(n+1)$ for each n.A set or sequence S is one of *disjoint rays* in G if every element is a ray in G and no two such rays have an element (i.e. vertex) in common. *G has arbitrarily many disjoint rays* if for every $k \in \mathbb{N}$ there is a sequence of disjoint rays of length k.

Theorem [IRT] (Halin [1965]): If a graph has arbitrarily many disjoint rays it has an infinite sequence $\langle R_n \rangle$ of disjoint rays.

THAs and Reductions

This sounds like a compactness result and so should be provable in ACA_0.

However, BGS show that it and several analogs including IRT_{XY} are THAs. (IRT_{XY} is IRT with the graph undirected or directed ($X = U$ or $X = D$) and the rays being vertex or edge disjoint ($Y = V$ or $Y = E$).

BGS investigated the reverse mathematical strength of these THAs as well as many other variations. In particular, they provided several reductions in RCA_0 (or RCA_0 with more induction) among many of these variations.

Locally Finite Graphs

One especially unusual phenomena has appeared in our investigations.

We were surprised to find what seemed to be a reduction we had missed in a much more recent paper of Bowler, Carmesian and Pott [2015]. They point out that IRT_{UE} follows from IRT_{US}.
In their reduction they sketch a proof of the following fact:

Proposition LF: If a graph G has arbitrarily many edge disjoint rays then it contains a locally finite graph $\hat{G}$ (each vertex is in only finitely many edges) with arbitrarily many edge disjoint rays.

The analogous statements LF_{XY} for directed graphs and/or vertex disjoint rays also hold.

The IRT_{XY} versions of Halin's theorem restricted to just locally finite graphs turn out to each be equivalent to ACA_0 (over RCA_0).

Thus each version LF_{XY} of the Proposition implies the corresponding version of Halin's theorem and so, at least over ACA_0, must be a THA.

But Not THA

The LF_{XY} themselves, however, are not THAs. Indeed, even together, they are highly conservative over RCA_0 and so (over RCA_0) they do not imply ACA_0 (or even WKL_0 or DNR_0). Moreover, even with the addition of WKL_0 they do not imply ACA_0.

So, even standard models of these facts about locally finite graphs are not necessarily closed under the Turing jump. However, if they are, they are closed under hyperarithmetic reductions as well.

We also have many variations on both classical and Halin type THAs which are almost THAs, i.e. when combined with ACA_0 they are THAs but on their own they are very weak in all the above senses and more.

ATHA

Definition: A sentence (theory) T is an *almost theorem (theory) of hyperarithmetic analysis (ATHA)* if $T + ACA_0$ is a THA but $T \nvdash ACA_0$ (and so is not itself a THA).

Most of our other examples of ATHAs are generated by weakening conclusions of known THAs to allow in some way for finitely many mistakes. In the following list Φ is arithmetical.

Σ^1_1-AC$_0$: $\forall n \exists X \Phi(n, X) \rightarrow \exists X \forall n \Phi(n, X^{[n]})$.
Σ^1_1-AC$_0^*$: $\forall n \exists X \Phi(n, X) \rightarrow \exists X \forall n (\exists Y (Y =^* X^{[n]} \ \& \ \Phi(n, Y)))$.
Σ^1_1-AC$_0^-$: $\forall n \exists X \Phi(n, X) \rightarrow \exists X \forall n \exists m \Phi(n, X^{[m]})$.
Σ^1_1-AC$_0^{-*}$: $\forall n \exists X \Phi(n, X) \rightarrow \exists X \forall n \exists m \exists Y (Y =^* X^{[m]} \ \& \ \Phi(n, Y))$.

All of these variants of Σ^1_1-AC$_0$ are equivalent over ACA$_0$ but all except Σ^1_1-AC$_0$ itself are weak as above. Similar variations of w-Σ^1_1-AC$_0$, f-Σ^1_1-AC$_0$ have the same status as do ones for the IRT$_{XY}$ and related principles,

Conservation Classes

We also prove many new types of conservation results for various classes of formulas for all these principles and others.

Definition: Each of our classes Γ of formulas consists of a base class which includes the quantifier free formulas and is then closed under conjunction ($\wedge$), disjunction ($\vee$), first order quantification ($\forall x$ and $\exists x$ for number variables) and universal second order quantification ($\forall X$ for set variables).

The G-Π^1_1 *class of formulas* has only the quantifier free ones in its base.

The G-r-Π^1_2 *class of formulas* also has in its base all formulas which are of the form $\exists Y \Theta(Y)$ where Θ is Σ^0_3.

The *G-Tanaka class of formulas* instead adds to the base class all formulas of the form $\exists! Y \Phi(Y)$ for arithmetic Φ.

The *G-r-Tanaka class* also includes in its base all formulas of the form $\exists! Y \exists Z \Psi(\bar{x}, Y, Z)$ with Ψ a Σ^0_3 formula.

For a class Γ of formulas, we say a theory T is Γ*-conservative* if, for every sentence $\varphi \in \Gamma$, $T \vdash \varphi \rightarrow RCA_0 \vdash \varphi$.

Tree Notions of Forcing

We define some quite general classes of forcings that have strong preservation properties over RCA_0 and can be iterated to prove conservation results for the class Γ defined above.

Definition: A notion of forcing $\mathcal{P} = \langle P, \leq \rangle$ is a *tree forcing (t-forcing)* if the following hold:

1. Conditions in $\mathcal{P}$ are of the form $\langle \tau, T \rangle$ where $T \in S(N)$ is a subtree of $N^{<N}$ (i.e. a subset of $N^{<N}$ in $\mathcal{N}$ closed under initial segments with respect to $\subseteq$) and τ is comparable with every $\sigma \in T$. The root of T is taken to be the empty string. The *stem* of T is defined to be the longest string comparable with every element of T.
2. If $\langle \tau', T' \rangle \leq \langle \tau, T \rangle$ then $\tau' \supseteq \tau$ and $T' \subseteq T$.
3. For every $n \in N$ the class $\{\langle \tau, T \rangle \mid |\tau| \geq n\}$ is dense in $\mathcal{P}$, i.e. $(\forall \langle \tau, T \rangle \in \mathcal{P})(\exists \langle \tau', T' \rangle)(\langle \tau', T' \rangle \leq \langle \tau, T \rangle \ \& \ |\tau'| \geq n)$.

Effective Tree Notions of Forcing

Definition: A tree notion of forcing $\mathcal{P}$ is an *effective tree forcing (et-forcing)* if, for every $\langle \tau, T\rangle \in \mathcal{P}$, the class $Ext(\langle \tau, T\rangle) = \{\tau' | (\exists T')(\langle \tau', T'\rangle \leq \langle \tau, T\rangle)\}$ is boldface Σ^0_1 i.e. there is an $A \in S(N)$ such that $Ext(\langle \tau, T\rangle)$ is $\Sigma^0_1(A)$ (over N).

Definition: An et-forcing $\mathcal{P}$ is *uniform (a uet-forcing)* if, for every condition $\langle \tau, T\rangle$, every $\rho, \sigma \in Ext(\langle \tau, T\rangle)$ with $|\rho| = |\sigma|$, and every $\langle \rho'', R''\rangle \leq \langle \rho', R'\rangle \leq \langle \tau, T\rangle$ with $\rho \subseteq \rho'$, $\langle \rho''_\sigma, R''_\sigma\rangle \leq \langle \rho'_\sigma, R'_\sigma\rangle \leq \langle \tau, T\rangle$. For technical convenience we also require that if $\langle \tau, T\rangle \in \mathcal{P}$ and the stem of T is some $\sigma \supset \tau$ then $\langle \rho, T\rangle \leq \langle \tau, T\rangle$ whenever $\sigma \supseteq \rho \supseteq \tau$. Note: For $\sigma \in T$, $T_\sigma = \{\mu_\sigma | \mu \in T\}$ where $\mu_\sigma(i) = \sigma(i)$ for $i < |\sigma|$ and $\mu_\sigma(i) = \mu(i)$ for $i \geq |\sigma|$.

Ignore this last definition. Uniformity roughly guarantees that in any condition $\langle \tau, T\rangle$ if σ and ρ extend τ and are of the same length then the subtrees of T above σ and ρ are the same.

Preservation Properties

Common examples of uet-forcings are Cohen, Mathias and Silver forcings and many variations. The usual versions of Laver and Sacks forcing are et but not uniform. Sacks forcing can be made so by using "uniform" trees as in Lerman [1983, VI.2.3]. Similar adjustments can be made to Laver forcing. These forcings have many preservation properties:

Theorem: If $\mathcal{P}$ is an et-forcing over a countable model $\mathcal{N}$ of RCA_0 with a suitably chosen countable collection $\mathcal{D}$ of dense sets, then we have the following preservation type results:

1. If G is $\mathcal{P}$-generic for $\mathcal{D}$, then $\mathcal{N}[G] \vDash \mathrm{RCA}_0$.
2. If G is $\mathcal{P}$-generic for $\mathcal{D}$ and $R \in S(\mathcal{N})$ is a subtree of $N^{<N}$ with no branch in $S(\mathcal{N})$, then it has none in $\mathcal{N}[G]$.
3. For any $\{C_i | i \in \omega\}$ with $C_i \subseteq N$ and $C_i \notin S(\mathcal{N})$ for every $i \in \omega$, there is a $\mathcal{D}$ such that, for any $\mathcal{D}$-generic G, no $C_i \in \mathcal{N}[G]$.

Conservation Consequences

We prove that these notions of forcing can be used to derive conservation results for each of our classes for theories T for which such forcings supply the witnesses required by the axioms/principles of T and so iterations can construct models of T.
All of our proofs have the same general format.

For the sake of a contradiction, we assume that there is a sentence $\Lambda \in \Gamma$ such that $T \vdash \Lambda$ and a countable model $\mathcal{N} \vDash \neg\Lambda$ of RCA_0.

We then construct, by iterated forcing, a model $\mathcal{N}_\infty$ of T.

If we can also guarantee that $\mathcal{N}_\infty \vDash \neg\Lambda$, we have proven Γ-conservativity.

Typically, the theories T consist (in addition to RCA_0) of Π^1_2 axioms of the form $\forall X(\Phi(X) \rightarrow \exists Y \Psi(X, Y))$ with Φ and Ψ arithmetic where we can add a witness Y for any instance X by forcing. Here, starting with any $\mathcal{N} \vDash RCA_0$, a careful ω length iteration produces a model $\mathcal{N}_\omega$ of T.

Theories

The crucial point for Π^1_2 axioms is that the facts being witnessed are arithmetic.

As forcing does not change the first order part of the ground model, absoluteness applies to show that solutions remain solutions and all instances in the final model are instances when they appear in the iteration.

The only new addition is that we move from the classical class (e.g. Π^1_1) to the generalized one (G-Π^1_1) by an induction argument on formulas.

Our ATHAs and many stronger principles with the same conservation properties are not Π^1_2 and additional arguments are needed.

Conservation and Preservation for ATHAs

For example, in the LF_{XY}, Φ and Ψ are of the form $\forall n \exists Z \Theta$ with Θ arithmetic (saying Z is a sequence of disjoint rays of length n).
While the added solutions remain solutions in $\mathcal{N}_\omega$, we may have new instances that did not seem to be instances at any point along the way: The required witnesses Z for some X may appear cofinally in the iteration. So $\mathcal{N}_\omega$ may not be a model of $\mathsf{LF}_{\mathrm{XY}}$.

The natural plan here is to continue the iteration to length ω_1.Then any witnesses for an X appearing in $\mathcal{N}_{\omega_1}$ must all appear at some countable stage of the length ω_1 iteration.

So, as we proceed carefully, we have a solution added at some point as well.

Theorem: For each of the $\mathrm{LF}_{\mathrm{XY}}$ there are uet-forcings that add solutions for any instance. Thus all of them together are G-r-Tanaka (and so G-Tanaka, G-r-Π^1_2 and G-Π^1_1) conservative over RCA_0. As over ACA_0 each implies $\mathrm{IRT}_{\mathrm{XY}}$ which is a THA, each of them is an ATHA.

Conservation Results for Stronger Theories

We also consider variations of the standard array of strong choice principles that provide ones that are Γ-conservative for all our classes but very strong over ACA_0.

Definition: Σ^1_{n+1}-AC*: $\forall A[\forall n\exists X\Phi(A,n,X) \to \exists Y\forall n\exists\sigma\Phi(A,n,Y^{[n]}_\sigma)]$, for Φ Π^1_n. (Note: $Y_\sigma(i)=\sigma(i)$ for $i<|\sigma|$ and $Y_\sigma(i)=Y(i)$ for $i\geq|\sigma|$.)
Σ^1_{n+1}-AC$^-$: $\forall A[\forall n\exists X\Phi(A,n,X) \to \exists Y\forall n\exists m\Phi(A,n,Y^{[m]})]$ for Φ Π^1_n.
Σ^1_∞-AC*: $\forall n(\Sigma^1_n$-AC$^*)$ and Σ^1_∞-AC$^-$: $\forall n(\Sigma^1_n$-AC$^-)$.

Theorem: $\forall n\in\omega$, $\mathrm{RCA}_0 \vdash \Sigma^1_{n+1}\text{-}\mathrm{AC}_0 \to \Sigma^1_{n+1}\text{-}\mathrm{AC}^* \to \Sigma^1_{n+1}\text{-}\mathrm{AC}^-$ and $\mathrm{ACA}_0 \vdash \Sigma^1_{n+1}\text{-}\mathrm{AC}^-_0 \to \Sigma^1_{n+1}\text{-}\mathrm{CA}_0$. So over ACA_0 all of $\Sigma^1_{n+1}\text{-}\mathrm{AC}^*$, $\Sigma^1_{n+1}\text{-}\mathrm{AC}^-$ and $\Sigma^1_{n+1}\text{-}\mathrm{CA}_0$ are equivalent for each n as are $\Sigma^1_\infty\text{-}\mathrm{AC}^*$, $\Sigma^1_\infty\text{-}\mathrm{AC}^-$ and $\Sigma^1_\infty\text{-}\mathrm{CA}_0$.

Theorem: $\Sigma^1_\infty\text{-}\mathrm{AC}^*_0$ and so $\Sigma^1_\infty\text{-}\mathrm{AC}^-_0$ and all the $\Sigma^1_{n+1}\text{-}\mathrm{AC}^*$ and $\Sigma^1_{n+1}\text{-}\mathrm{AC}^-$ are Γ-conservative for all our classes Γ.

ATHAs and Stronger Theories

In particular, Σ^1_1-AC^*_0, Σ^1_1-AC^-_0, Σ^1_∞-AC^*_0 and Σ^1_∞-AC^- are highly conservative over RCA_0 but over ACA_0 each of the first pair are equivalent to Σ^1_1-AC_0 (and so are ATHAs) and each of the second pair are equivalent to Σ^1_∞-AC_0 and so stronger than full second order arithmetic.

Some earlier conservation results for some of the theories covered here are in work by Yamazaki [2000], Kihara [2008] or by Tanaka, Montalbán and Yamazaki as reported in Yamazaki [2009] and in Yokoyama [2009].

The proof of the next to last Theorem is combinatorial and proceeds by induction on n. The last Theorem is proven by first providing uet-forcings that add solutions for Σ^1_∞-AC^*_0.

Now Σ^1_∞-AC^*_0 has both hypotheses/instances $\Phi(X)$ and conclusions/solutions $\Psi(X, Y)$ of arbitrary complexity. Thus we need another idea to guarantee that adding what looks like a solution stays a solution in $\mathcal{N}_{\omega_1}$ as well as a procedure that makes sure we handle everything that is an instance in $\mathcal{N}_{\omega_1}$ along the way.

A CLUB of Elementary Submodels

The crucial idea here is that if we do an ω_1 iteration, then for a closed unbounded set of $\lambda < \omega_1$, $\mathcal{N}_\lambda$ will be an elementary submodel of $\mathcal{N}_{\omega_1}$. Thus if we carefully handle everything that looks like an instance in any $\mathcal{N}_\lambda$ and supply something that looks like a solution (over $\mathcal{N}_\lambda$) all will be well at the end.

Answering Another Open Question?

We view these results on variants of choice principles and the previous ones on ATHAs that are equivalent to known THAs over ACA_0 as supplying answers to a question raised by Hirschfeldt and repeated in Montalbán's Open Questions in Reverse Mathematics [2011].

He asked for examples of principles that are distinct over RCA_0 (or even RCA) but equivalent over some stronger (natural) theory.

Our results provide an ample list of many pairs of principles that are very different over RCA_0 but equivalent over ACA_0. The most mathematically natural ones are the IRT_{XY} and the corresponding LF_{XY}. It could well be argued that the weak ones should really be seen as the same as their strong counterparts in an analysis that works over ACA_0 rather than RCA_0.