Department of Mathematics, Computer Science, Physics
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Ill-founded orders and Weihrauch degrees

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Joint work with Jun Le Goh and Arno Pauly

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What about the uniform computational content?

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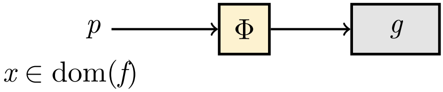

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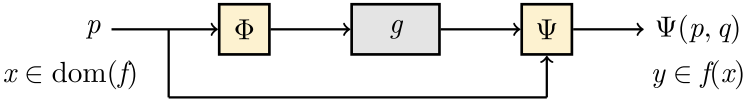
- Given a name p for $x \in \mathrm{dom}(f)$, $\Phi(p)$ is a name for $z \in \mathrm{dom}(g)$
- Given a name q for $w \in g(z)$, $\Psi(p, q)$ is a name for $y \in f(x)$



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$$\{\langle a, b\rangle \in \mathbb{N} : a \leq_L b\}$$

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We define $\mathsf{DS} :\subseteq \mathrm{LO} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ as

$$\mathsf{DS}(\leq_L) := \{x \in \mathbb{N}^{\mathbb{N}} : (\forall i)(x(i+1) <_L x(i))\}$$

with $\mathrm{dom}(\mathsf{DS}) := \mathrm{LO} \setminus \mathrm{WO}$.

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Similarly to DS, consider the represented space $(\mathrm{QO}, \delta_{\mathrm{QO}})$ of countable quasi orders.

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$$\mathsf{BS} :\subseteq \mathrm{QO} \rightrightarrows \mathbb{N}^{\mathbb{N}} := \preceq_Q \mapsto \{x \in \mathbb{N}^{\mathbb{N}} : (\forall i < j)(x(i) \not\preceq_Q x(j))\}$$

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Theorem (Folklore?)

$\mathsf{DS} \equiv_{\mathrm{W}} \mathsf{BS}$

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$\mathsf{C}_{\mathbb{N}^\mathbb{N}}$: given an ill-founded tree $T \subset \mathbb{N}^{<\mathbb{N}}$, compute a path through T

$\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$: restriction of $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ to trees with a single path

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$\mathsf{DS} \not\le_W \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$: $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ always has an hyperarithmetic solution (in the input). □

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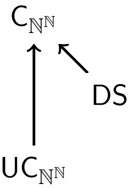

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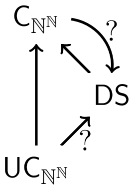

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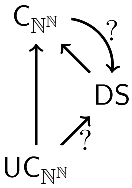

Spoiler: no to both of the questions.

First-order part of a problem

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Recently introduced by (Dzhafarov, Solomon, Yokoyama).

Let $\mathcal{FO}$ be the set of problems with codomain $\mathbb{N}$.

For $f :\subseteq X \rightrightarrows Y$, the *first-order part of f* is a problem ${}^1 f \in \mathcal{FO}$ s.t.

$${}^1 f \equiv_{\mathrm{W}} \max_{\leq_{\mathrm{W}}} \{f_0 \in \mathcal{FO} \,:\, f_0 \leq_{\mathrm{W}} f\}$$

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Obviously

$$ f \leq_{\mathrm{W}} g \Rightarrow {}^1 f \leq_{\mathrm{W}} {}^1 g $$

$$ {}^1 f \not\equiv_{\mathrm{W}} {}^1 g \Rightarrow f \not\equiv_{\mathrm{W}} g $$

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$$\mathrm{Det}_Y(f) \equiv_{\mathrm{W}} \max_{\leq_{\mathrm{W}}} \{f_0 :\subseteq X \to Y : f_0 \leq_{\mathrm{W}} f\}$$

If $Y = \mathbb{N}^{\mathbb{N}}$ we just write $\mathrm{Det}(f)$.

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This is a degree-theoretic operator, hence

$$\mathrm{Det}_Y(f) \not\equiv_{\mathrm{W}} \mathrm{Det}_Y(g) \Rightarrow f \not\equiv_{\mathrm{W}} g$$

First-order part of DS

Define $\mathbf{\Pi}^1_1\text{–}\mathsf{Bound} :\subseteq \mathbf{\Pi}^1_1(\mathbb{N}) \rightrightarrows \mathbb{N}$ as

$$\mathbf{\Pi}^1_1\text{–}\mathsf{Bound}(A) := \{b \in \mathbb{N} \,:\, (\forall n \in A)(n \le b)\}$$

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Since $^1\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathbf{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}$ (easy) and $\mathbf{\Pi}^1_1-\mathsf{Bound} <_{\mathrm{W}} \mathbf{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}$ (Anglès D'Auriac, Kihara), we have

Corollary (Goh, Pauly, V.)

$\mathsf{C}_{\mathbb{N}^\mathbb{N}} \not\le_{\mathrm{W}} \mathsf{DS}$

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Corollary (Goh, Pauly, V.)

$\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}} \not\leq_{\mathrm{W}} \mathsf{DS}$

Other characterizations

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If $f :\subseteq X \rightrightarrows \mathbb{N}$ *is s.t.* $f(x)$ *is finite for every* $x \in \mathrm{dom}(f)$ *then then*

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If $f :\subseteq X \rightrightarrows k$, *then* $f \leq_{\mathrm{W}} \mathsf{DS} \iff f \leq_{\mathrm{W}} \mathsf{RT}^1_k$.

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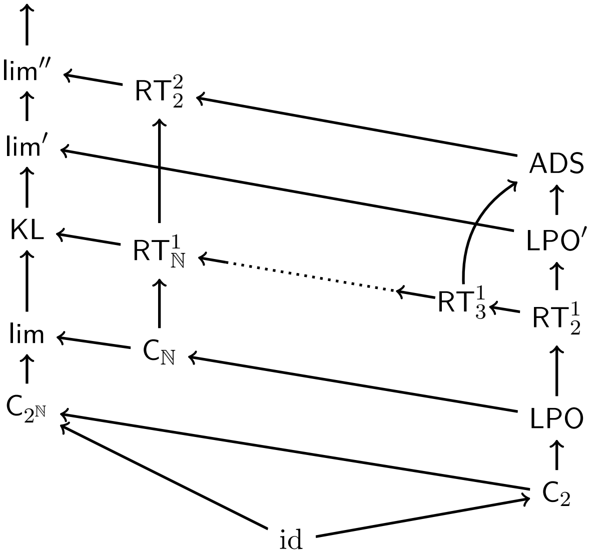
Theorem (Goh, Pauly, V.)

If $f :\subseteq X \to k$, then $f \leq_{\mathrm{W}} \mathsf{DS} \iff f \leq_{\mathrm{W}} \mathsf{lim}_k$.

Overview of the Weihrauch lattice



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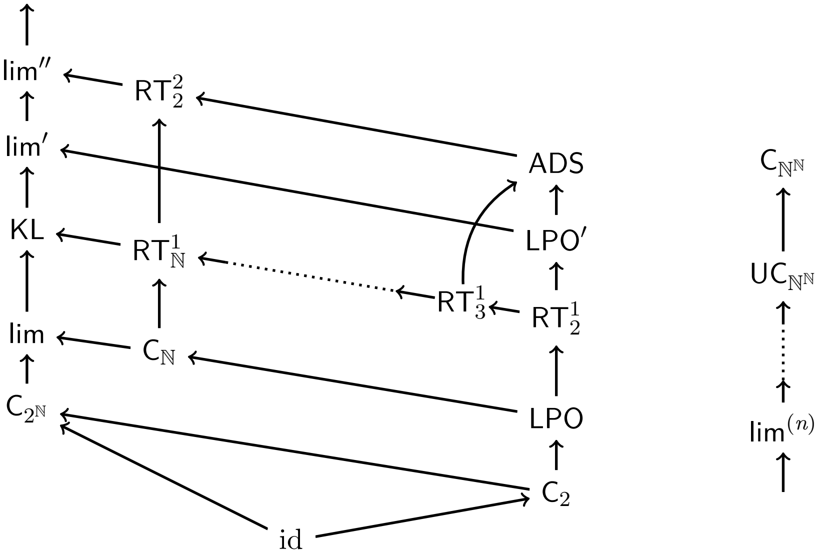

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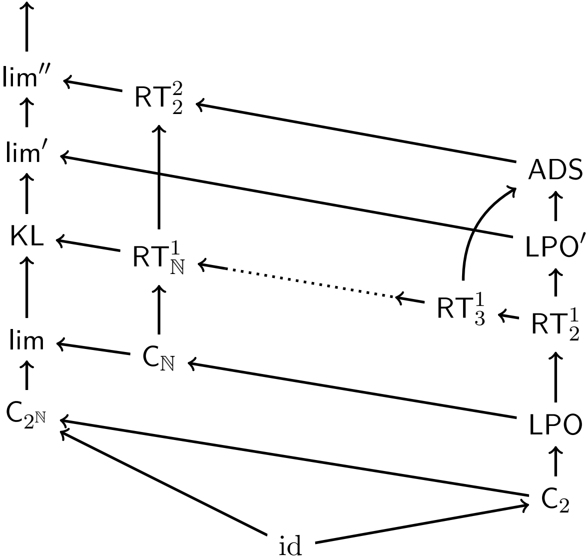

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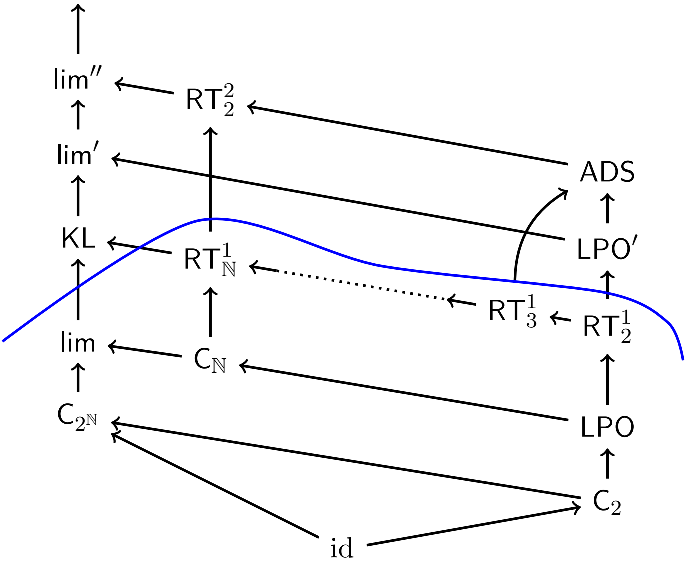

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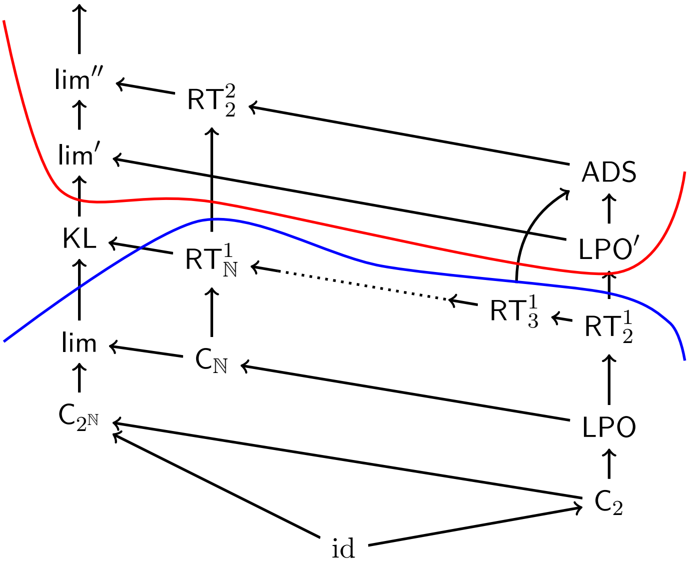

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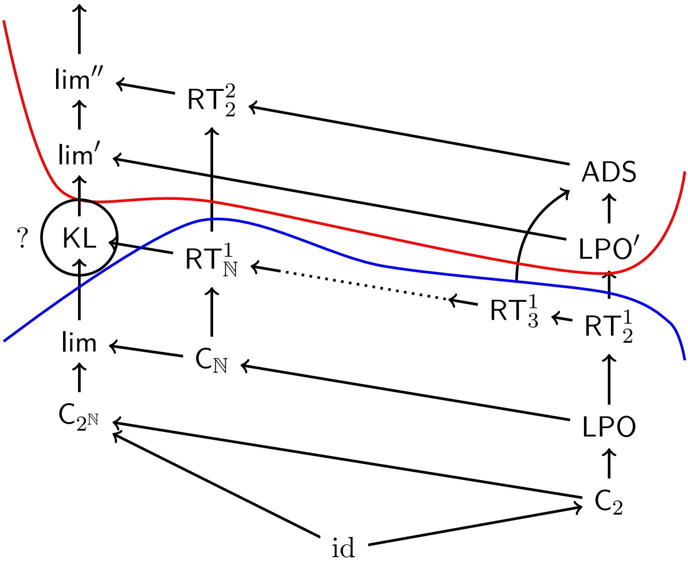

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Let $\mathbf{\Gamma} \in \{\mathbf{\Sigma}^0_k, \mathbf{\Pi}^0_k, \mathbf{\Delta}^0_k, \mathbf{\Sigma}^1_1, \mathbf{\Pi}^1_1, \mathbf{\Delta}^1_1\}$ and consider the represented spaces $\mathbf{\Gamma}(\mathrm{LO})$ and $\mathbf{\Gamma}(\mathrm{QO})$.

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We define $\mathbf{\Gamma}\text{-}\mathsf{DS} :\subseteq \mathbf{\Gamma}(\mathrm{LO}) \rightrightarrows \mathbb{N}^{\mathbb{N}}$ and $\mathbf{\Gamma}\text{-}\mathsf{BS} :\subseteq \mathbf{\Gamma}(\mathrm{QO}) \rightrightarrows \mathbb{N}^{\mathbb{N}}$ as

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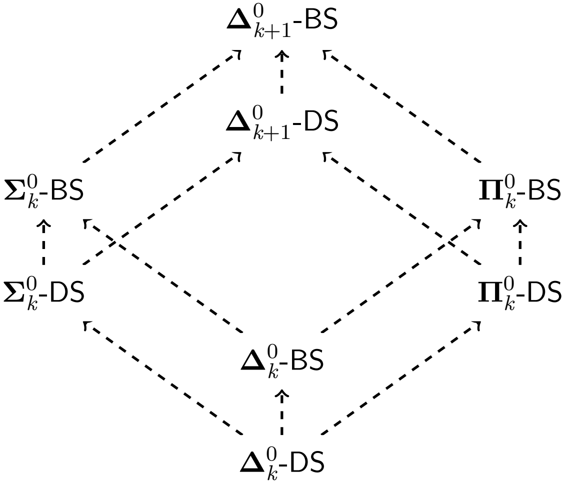
This creates a hierarchy of DS-like problems.

$$\mathbf{\Gamma}\text{-}\mathsf{DS} \leq_{\mathrm{W}} \mathbf{\Gamma}\text{-}\mathsf{BS}$$
$$\mathbf{\Gamma} \subset \mathbf{\Gamma}' \implies \mathbf{\Gamma}\text{-}\mathsf{DS} \leq_{\mathrm{W}} \mathbf{\Gamma}'\text{-}\mathsf{DS}$$
$$\mathbf{\Gamma} \subset \mathbf{\Gamma}' \implies \mathbf{\Gamma}\text{-}\mathsf{BS} \leq_{\mathrm{W}} \mathbf{\Gamma}'\text{-}\mathsf{BS}$$

An arithmetic DS hierarchy



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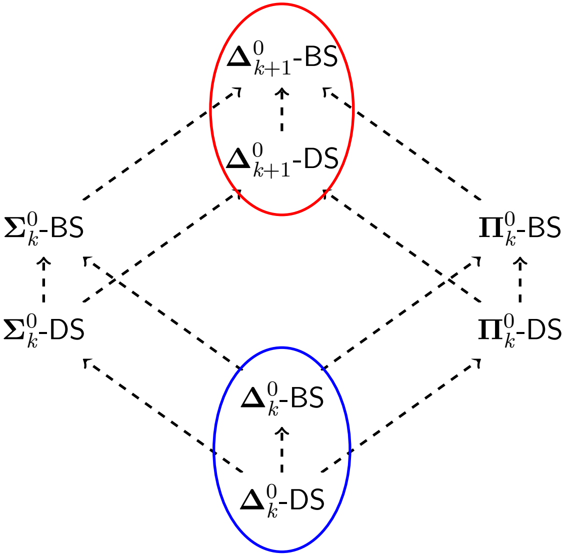

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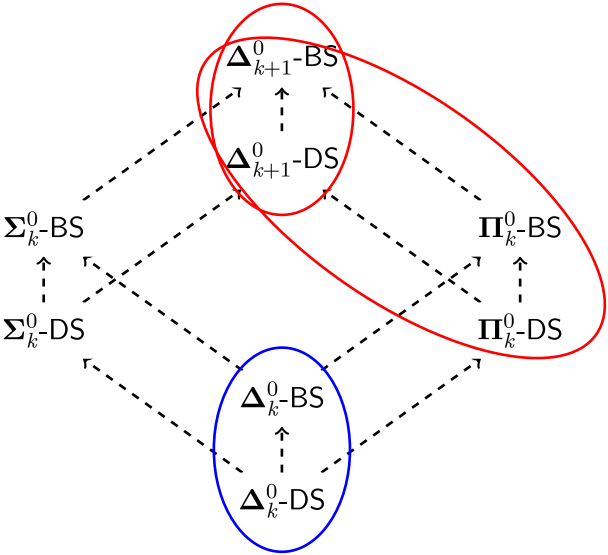

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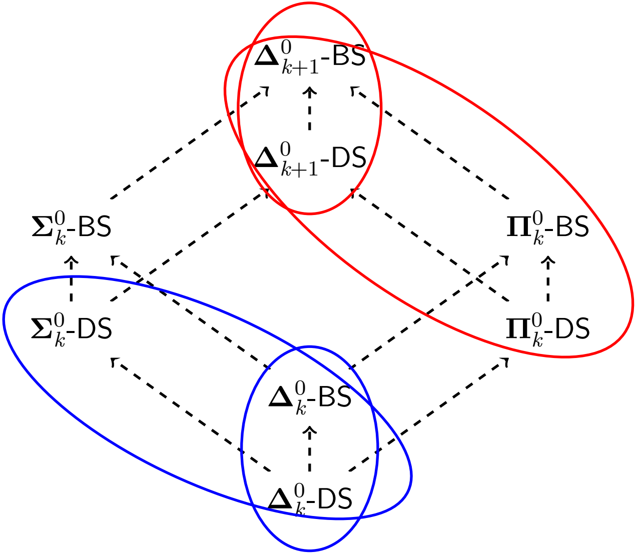

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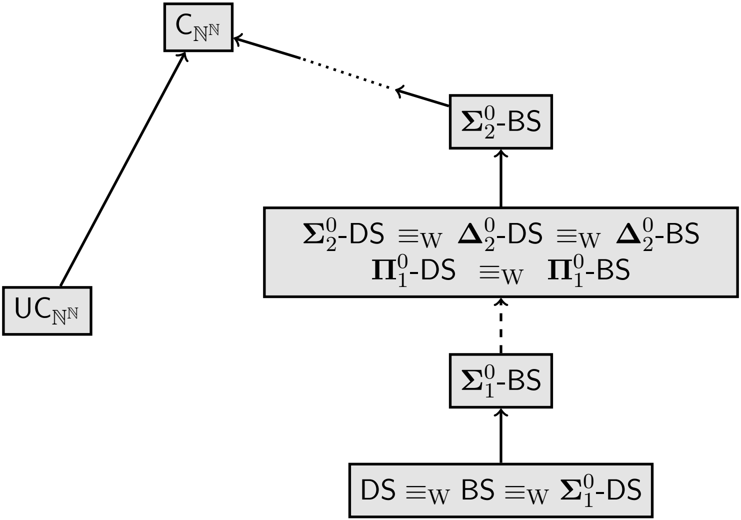

Beyond the arithmetic classes

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$\boldsymbol{\Delta}^1_1$-DS is the first level of the DS hierarchy that computes $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$.

Theorem (Goh, Pauly, V.)

$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} \boldsymbol{\Delta}^1_1\text{-}\mathsf{DS} \equiv_{\mathrm{W}} \mathsf{DS} * \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$

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However it does not reach $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$:

Theorem (Goh, Pauly, V.)

${}^1\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_{\mathbb{N}} \not\leq_{\mathrm{W}} \boldsymbol{\Delta}^1_1\text{-}\mathsf{DS}$, *and hence* $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \not\leq_{\mathrm{W}} \boldsymbol{\Delta}^1_1\text{-}\mathsf{DS}$.

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However it does not reach $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$:

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$^1\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_\mathrm{W} \boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N} \not\leq_\mathrm{W} \boldsymbol{\Delta}^1_1\text{-}\mathsf{DS}$, *and hence* $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \not\leq_\mathrm{W} \boldsymbol{\Delta}^1_1\text{-}\mathsf{DS}$.

To compute $\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}$ we need to climb a step higher

Theorem (Goh, Pauly, V.)

$\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N} <_\mathrm{W} \widehat{\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}} \leq_\mathrm{W} \boldsymbol{\Sigma}^1_1\text{-}\mathsf{DS}$

Beyond the arithmetic classes

$\mathbf{\Delta}^1_1$-DS is the first level of the DS hierarchy that computes $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$.

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Does $\mathbf{\Sigma}^1_1$-DS reach the level of $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$?

$\widehat{\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_{\mathbb{N}}}$ and ATR_2

(Kihara, Marcone, Pauly) showed that $\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}} <_{\mathrm{W}} \widehat{\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_{\mathbb{N}}} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$.

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ATR_2: Given a linear order L, produce either a L-descending sequence or a jump hierarchy supported on L (and a bit indicating which one). $\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}} <_{\mathrm{W}} \mathsf{ATR}_2 <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$

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$\mathsf{ATR}_2 \not\leq_\mathrm{W} \widehat{\mathbf{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}}$ *and hence* $\widehat{\mathbf{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}} <_\mathrm{W} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$

$\boldsymbol{\Sigma}_1^1$-DS and ATR$_2$

Extending the technique used by (Anglès D'Auriac, Kihara), we can prove a stronger result

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The proof is based on the following result

Theorem (Goh)

Let **wf** *be the set of indices for well-founded linear orders, and let* **hds** *be the set of indices for linear orders with HYP descending sequences.*
Any Σ^1_1 *set separating* **wf** *and* **hds** *is complete.*

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The proof is based on the following result

Theorem (Goh)

Let **wf** *be the set of indices for well-founded linear orders, and let* **hds** *be the set of indices for linear orders with HYP descending sequences.*
Any Σ_1^1 *set separating* **wf** *and* **hds** *is complete.*

We show that a reduction $\mathsf{ATR}_2 \leq_{\mathrm{W}} \boldsymbol{\Sigma}_1^1\text{-}\mathsf{DS}$ would yield a Δ_1^1 set separating **wf** and **hds**.

Beyond the arithmetic classes

The problems $\mathbf{\Sigma}^1_1$-BS and $\mathbf{\Pi}^1_1$-DS are much stronger

$\mathbf{\Pi}^1_1$-CA is the analogue of $\mathbf{\Pi}^1_1-\mathrm{CA}_0$: given a sequence $(T_i)_{i\in\mathbb{N}}$, produce $x \in 2^{\mathbb{N}}$ s.t. $x(i) = 1 \iff [T_i] = \emptyset$

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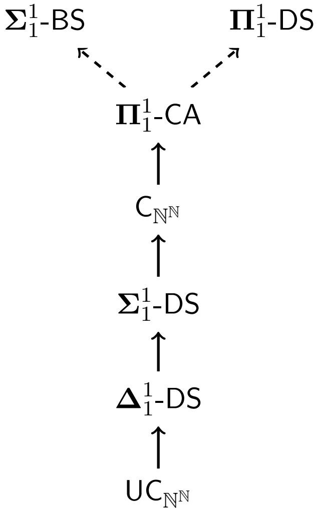
Theorem (Goh, Pauly, V.)

$\mathbf{\Pi}^1_1$-$\mathsf{CA} \le_{\mathrm{W}} \mathbf{\Sigma}^1_1$-$\mathsf{BS}$ *and* $\mathbf{\Pi}^1_1$-$\mathsf{CA} \le_{\mathrm{W}} \mathbf{\Pi}^1_1$-$\mathsf{DS}$

$\mathbf{\Sigma}^1_1$-BS and $\mathbf{\Pi}^1_1$-DS can be used to compute the leftmost path of an ill-founded tree.

Beyond the arithmetic classes



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