Generalized Computable Numberings and Degree Spectra of Hereditarily Countable Families

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Basic Definitions

Numbering of a countable set S is a surjective mapping $\nu : \mathbb{N} \to S$.

Let $H(S)$ be the set of all numberings of S. Let $\nu_0, \nu_1 \in H(S)$.

Definition

We say that ν_0 is reducible to ν_1 ($\nu_0 \leqslant \nu_1$) if $\nu_0 = \nu_1 \circ f$ for some computable function f. Numberings ν_0 and ν_1 are called equivalent ($\nu_0 \equiv \nu_1$) if $\nu_0 \leqslant \nu_1$ and $\nu_1 \leqslant \nu_0$.

Definition

A numbering ν of a countable family $\mathcal{S} \subseteq 2^{\mathbb{N}}$ is computable, if the set $G_\nu = \{\langle x, y \rangle : y \in \nu(x)\}$ is c.e. (or, equivalently, there is a computable function h such that $\nu(x) = W_{h(x)}$). In this case, the family $\mathcal{S}$ is said to be also computable.

Approach of Goncharov-Sorbi (1997)

Let $\mathscr{C}$ be a family of constructive objects described by elements of some language $\mathscr{L}$. Suppose that the language $\mathscr{L}$ is equipped with Gödel numbering γ. Let I be an interpretation of the expressions from $\mathscr{L}$, i.e. let $I : \mathscr{L} \to \mathscr{C}$ be any surjective mapping.

Examples

1. Let $\mathscr{C}$ be the family of all Σ^0_1-subclasses of the Cantor space $2^{\mathbb{N}}$, $\mathscr{L}$ the class of all c.e. subsets of $2^{<\mathbb{N}}$. Then we can define $I(S) = \bigcup_{\sigma \in S}\{Z : \sigma \prec Z\}$.

2. Let $\mathscr{C}$ be the set of all left-c.e. reals, $\mathscr{L}$ the set of all pairs $\langle W, q\rangle$, where $W \subseteq \mathbb{Q}$ is a c.e. set and $(-\infty, q) \cap W \neq \emptyset$. Then we can define $I(W, q) = \sup\{r \in W : r < q\}$.

Approach of Goncharov-Sorbi (1997)

Let $\mathscr{C}$ be some class of objects, $\mathscr{L}$ a language describing the elements of $\mathscr{C}$, γ a Gödel numbering of $\mathscr{L}$, I an interpretation of $\mathscr{L}$ in $\mathscr{C}$.

A numbering $\nu : \mathbb{N} \longrightarrow \mathcal{S} \subseteq \mathscr{C}$ is called computable numbering (relative to the interpretation I) if there exists a computable function f s.t. $\nu(n) = I(\gamma_{f(n)})$ for each $n \in \mathbb{N}$.

Let $\mathrm{Com}_I^{\mathscr{L}}(\mathcal{S})$ be the class of all such numberings.

The quotient structure $\mathscr{R}_I^{\mathscr{L}}(\mathcal{S}) = \langle \mathrm{Com}_I^{\mathscr{L}}(\mathcal{S})_{/\equiv}; \leqslant \rangle$ is the Rogers semilattice of the family $\mathcal{S}$. Join in $\mathscr{R}_I^{\mathscr{L}}(\mathcal{S})$ is induced by the direct sum of numberings: $(\nu_0 \oplus \nu_1)(2x+i) = \nu_i(x)$, $i = 0, 1$.

Computable Numberings in the Arithmetical Hierarchy

Let $\mathscr{C}$ be the class Σ^0_{n+1}, $\mathscr{L}$ be the set of all Σ_{n+1}-formulas of arithmetics of a free variable x.

Let $I(\gamma_m) = \{a : \Omega \models \gamma_m[a]\}$, where Ω is the standard model of arithmetic.

Then a numbering ν of a family $\mathcal{S} \subseteq \mathscr{C}$ is called Σ^0_{n+1}-computable if there exists a computable function f s.t. $\nu(m) = \{a : \Omega \models \gamma_{f(m)}[a]\}$ for each $m \in \mathbb{N}$.

Let $\mathrm{Com}^0_{n+1}(\mathcal{S}) = \mathrm{Com}^{\mathscr{L}}_I(\mathcal{S})$ and $\mathscr{R}^0_{n+1}(\mathcal{S}) = \mathscr{R}^{\mathscr{L}}_I(\mathcal{S})$.

Theorem (Goncharov, Sorbi, 1997)

A numbering ν of a family $\mathcal{S}$ of Σ^0_{n+1}-sets is Σ^0_{n+1}-computable iff $G_\nu = \{\langle x, y\rangle : y \in \nu(x)\} \in \Sigma^0_{n+1}$.

The hereditarily countable families of rank 1 (1-families) are the countable subsets of $2^{\mathbb{N}}$.

The hereditarily countable families of rank $(n+1)$ ($(n+1)$-families) are the countable sets of n-families. The 2-families are also called the classes of families.

Let $\mathscr{C}$ be the class of all computable families, $\mathscr{L}$ be the class of all computable numberings. Let $I(\gamma_m) = \gamma_m(\mathbb{N})$.

Then a numbering ν of a 2-family $\mathfrak{S} \subseteq \mathscr{C}$ is called computable if there exists a computable function f s.t. $\nu(m) = \gamma_{f(m)}(\mathbb{N})$ (or, equialently, there is a computable function h s.t. $\nu(m) = \{W_{h(m,x)} : x \in \mathbb{N}\}$). In this case, the 2-family $\mathfrak{S}$ is also called computable.

Let $\mathscr{C}$ be the class of all computable n-families, $\mathscr{L}$ be the class of all their computable numberings. Let $I(\gamma_m) = \gamma_m(\mathbb{N})$.

Then a numbering ν of an $(n+1)$-family $\mathfrak{S} \subseteq \mathscr{C}$ is called computable if there exists a computable function f s.t. $\nu(m) = \gamma_{f(m)}(\mathbb{N})$.

Let $\mathrm{Com}^h_{n+1}(\mathfrak{S}) = \mathrm{Com}^{\mathscr{L}}_I(\mathfrak{S})$ and $\mathscr{R}^h_{n+1}(\mathfrak{S}) = \mathscr{R}^{\mathscr{L}}_I(\mathfrak{S})$.

If $\mathcal{S}$ is a computable family, then $\mathscr{R}^0_1(\mathcal{S}) \cong \mathscr{R}^h_2(\mathfrak{F}(\mathcal{S}))$, where $\mathfrak{F}(\mathcal{S}) = \{\{f \in \mathbb{N}^{\mathbb{N}} : f = x,\ x \in A\} : A \in \mathcal{S}\}$.

For a set $A \subseteq \mathbb{N}$, let
$\mathfrak{F}_0(A) = \{f \in \mathbb{N}^{\mathbb{N}} : \exists x \in A \forall y \, [f(y) = x + 1]\} \cup \{f \in \mathbb{N}^{\mathbb{N}} : f =^* 0\}$,
$\mathfrak{F}_{n+1}(A) = \{\mathfrak{F}_n(f) : f \in \mathfrak{F}_0(A)\}$.

For a family $\mathscr{B}$, let $\mathfrak{S}_n(\mathscr{B}) = \{\mathfrak{F}_n(A) : A \in \mathscr{B}\}$.

Theorem

Let $\mathscr{B}$ be a Σ^0_{n+2}-computable family. Then $\mathscr{R}^0_{n+2}(\mathscr{B}) \cong \mathscr{R}^h_{n+2}(\mathfrak{S}_n(\mathscr{B}))$. In particular, for $n = 0$ we have that $\mathscr{R}^0_2(\mathscr{B})$ is isomorphic to the Rogers semilattice $R^h_2(\mathfrak{S}_0(\mathscr{B}))$ of the class of computable functions $\mathfrak{S}_0(\mathscr{B})$.

For a class of families of computable functions $\mathfrak{H}$, let $C(\mathfrak{H}) = \{\mathscr{F}^c : \mathscr{F} \in \mathfrak{H}\}$, where $\mathscr{F}^c$ is the closure of $\mathscr{F}$ in the Baire space $\mathbb{N}^{\mathbb{N}}$.

Complutable Classes of Families of Total Functions

Let $\mathscr{A}$ and $\mathscr{B}$ be finite families of Σ^0_{n+1}-sets s.t. $\langle \mathscr{A}; \subseteq \rangle \cong \langle \mathscr{B}; \subseteq \rangle$.
Then $\mathscr{R}^0_{n+1}(\mathscr{A}) \cong \mathscr{R}^0_{n+1}(\mathscr{B})$ (see Ershov's monograph, 1977).

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Let $\mathfrak{F}$ and $\mathfrak{G}$ be finite classes of families of computable functions with $|\mathfrak{F}| = |C(\mathfrak{F})|$ and $|C(\mathfrak{G})| = 1$.

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Let $\mathfrak{F}$ and $\mathfrak{G}$ be finite classes of families of computable functions with $|\mathfrak{F}| = |C(\mathfrak{F})|$ and $|C(\mathfrak{G})| = 1$. Let $\mathscr{A}$ be a finite family of c.e. sets s.t. $\langle \mathscr{A}; \subseteq \rangle \cong \langle C(\mathfrak{F}); \subseteq \rangle$. Then $\mathscr{R}^0_1(\mathscr{A}) \cong \mathscr{R}^h_2(\mathfrak{F})$. In particular, the case is possible when $|\mathfrak{F}| > 1$ but $|\mathscr{R}^h_2(\mathfrak{F})| = 1$.

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If $\mathcal{S}$ is a finite family of Σ^0_2-sets s.t. $\langle \mathcal{S}; \subseteq \rangle \cong \langle \mathfrak{G}; \subseteq \rangle$, then $\mathscr{R}^0_2(\mathcal{S}) \cong \mathscr{R}^h_2(\mathfrak{G})$.

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Let $\mathfrak{F}$ and $\mathfrak{G}$ be finite classes of families of computable functions with $|\mathfrak{F}| = |C(\mathfrak{F})|$ and $|C(\mathfrak{G})| = 1$. Let $\mathscr{A}$ be a finite family of c.e. sets s.t. $\langle \mathscr{A}; \subseteq \rangle \cong \langle C(\mathfrak{F}); \subseteq \rangle$. Then $\mathscr{R}^0_1(\mathscr{A}) \cong \mathscr{R}^h_2(\mathfrak{F})$. In particular, the case is possible when $|\mathfrak{F}| > 1$ but $|\mathscr{R}^h_2(\mathfrak{F})| = 1$.

If $\mathcal{S}$ is a finite family of Σ^0_2-sets s.t. $\langle \mathcal{S}; \subseteq \rangle \cong \langle \mathfrak{G}; \subseteq \rangle$, then $\mathscr{R}^0_2(\mathcal{S}) \cong \mathscr{R}^h_2(\mathfrak{G})$.

Theorem (Goncharov, Sorbi, 1997)

Let $\mathcal{S}$ be a Σ^0_{n+2}-computable family with $|\mathcal{S}| > 1$. Then $\mathscr{R}^0_2(\mathcal{S})$ is infinite. In particular, the case is possible when $\langle \mathfrak{F}; \subseteq \rangle \cong \langle \mathfrak{G}; \subseteq \rangle$ but $\mathscr{R}^h_2(\mathfrak{F}) \ncong \mathscr{R}^h_2(\mathfrak{G})$.

Complutable Classes of Families of Total Functions

Let $\mathfrak{F}_0, \ldots, \mathfrak{F}_n$ be finite classes of families of computable functions s.t. $(\forall i, j \leqslant n)(\forall \mathcal{G} \in \mathfrak{F}_i)(\forall \mathcal{H} \in \mathfrak{F}_j)\, [\mathcal{G}^c = \mathcal{H}^c \Leftrightarrow i = j]$.

Complutable Classes of Families of Total Functions

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Let $\mathcal{R} = \{R_m : m \leqslant n\}$ be a family of c.e. sets such that $R_i \subseteq R_j \Leftrightarrow \mathcal{G}_i^c \subseteq \mathcal{G}_j^c$, where $\mathcal{G}_i \in \mathfrak{F}_i$, $\mathcal{G}_j \in \mathfrak{F}_j$.

Complutable Classes of Families of Total Functions

Let $\mathfrak{F}_0, \ldots, \mathfrak{F}_n$ be finite classes of families of computable functions s.t. $(\forall i, j \leqslant n)(\forall \mathcal{G} \in \mathfrak{F}_i)(\forall \mathcal{H} \in \mathfrak{F}_j)\,[\mathcal{G}^c = \mathcal{H}^c \Leftrightarrow i = j]$.

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Let $\mathcal{A}_i$ be a finite family of Σ_2^0-sets s.t. $\langle \mathcal{A}_i; \subseteq \rangle \cong \langle \mathfrak{F}_i; \subseteq \rangle$, $i \leqslant n$.

Complutable Classes of Families of Total Functions

Let $\mathfrak{F}_0, \dots, \mathfrak{F}_n$ be finite classes of families of computable functions s.t. $(\forall i, j \leqslant n)(\forall \mathcal{G} \in \mathfrak{F}_i)(\forall \mathcal{H} \in \mathfrak{F}_j)\,[\mathcal{G}^c = \mathcal{H}^c \Leftrightarrow i = j]$.

Let $\mathcal{R} = \{R_m : m \leqslant n\}$ be a family of c.e. sets such that $R_i \subseteq R_j \Leftrightarrow \mathcal{G}_i^c \subseteq \mathcal{G}_j^c$, where $\mathcal{G}_i \in \mathfrak{F}_i$, $\mathcal{G}_j \in \mathfrak{F}_j$.

Let $\mathcal{A}_i$ be a finite family of Σ^0_2-sets s.t. $\langle \mathcal{A}_i; \subseteq \rangle \cong \langle \mathfrak{F}_i; \subseteq \rangle$, $i \leqslant n$.

We define the family $\mathcal{S} = \{R_i \oplus A : A \in \mathcal{A}_i, i \leqslant n\}$ and the ideal $I(\mathcal{S}) = \{[x \mapsto W_{f(x)} \oplus W^{\emptyset'}_{g(x)}] \in \mathcal{R}^0_2(\mathcal{S}) : f, g \text{ are computable}\}$ of $\mathcal{R}^0_2(\mathcal{S})$.

Theorem

Let $\mathfrak{F} = \bigcup_{i \leqslant n} \mathfrak{F}_i$. Then $\mathcal{R}^h_2(\mathfrak{F}) \cong I(\mathcal{S})$.

Universal Numberings

A numbering $\nu \in \mathrm{Com}_I^{\mathcal{L}}(\mathcal{S})$ is universal if $\alpha \leqslant \nu$ for each $\alpha \in \mathrm{Com}_I^{\mathcal{L}}(\mathcal{S})$.

Theorem (Lachlan, 1964)

Any finite family of c.e. sets has a universal numbering.

Theorem (Badaev, Goncharov, Sorbi, 2003)

Let $\mathcal{S}$ be a finite family of Σ^0_{n+2}-sets. Then $\mathcal{S}$ has a universal numbering iff $\bigcap \mathcal{S} \in \mathcal{S}$.

Theorem

Let $\mathfrak{F}_0, \ldots, \mathfrak{F}_n$ be finite classes of families of computable functions s.t. $(\forall i, j \leqslant n)(\forall \mathcal{G} \in \mathfrak{F}_i)(\forall \mathcal{H} \in \mathfrak{F}_j)\,[\mathcal{G}^c = \mathcal{H}^c \Leftrightarrow i = j]$. Then $\mathfrak{F} = \bigcup_{i \leqslant n} \mathfrak{F}_i$ has a universal numbering iff $\forall i \leqslant n\,[\mathfrak{F}_i$ has a universal numbering$]$ iff $\forall i \leqslant n\,[\bigcap \mathfrak{F}_i \in \mathfrak{F}_i]$.

Universal Numberings

Theorem (Lachlan, 1964)

Let $\mathcal{S}$ be a computable family with a universal numbering. Then $\bigcup_x \alpha(x) \in \mathcal{S}$ for any computable numbering $\alpha : \mathbb{N} \xrightarrow{\text{onto}} \mathcal{S}' \subseteq \mathcal{S}$ such that $\alpha(0) \subseteq \alpha(1) \subseteq \alpha(2) \subseteq \dots$.

Theorem

1. Let $\mathfrak{F}$ be a computable class of total functions with a universal numbering. Then $(\bigcup_x \alpha(x))^c \in C(\mathfrak{F})$ for any computable numbering $\alpha : \mathbb{N} \xrightarrow{\text{onto}} \mathfrak{F}' \subseteq \mathfrak{F}$ such that $\alpha(0) \subseteq \alpha(1) \subseteq \alpha(2) \subseteq \dots$.

2. There are a computable class $\mathfrak{F}$ of total functions with a universal numbering and a computable numbering $\alpha : \mathbb{N} \xrightarrow{\text{onto}} \mathfrak{F}' \subseteq \mathfrak{F}$ such that $\alpha(0) \subseteq \alpha(1) \subseteq \alpha(2) \subseteq \dots$ and $\bigcup_x \alpha(x) \notin \mathfrak{F}$.

Cardinality and Latticeness of $\mathscr{R}^0_1(\mathcal{S})$

Questions (Ershov, 1967)

1. What can we say about the cardinalities of the Rogers semilattices?
2. When are they lattices?

Theorem (Khutoretskii, 1971)

If $|\mathscr{R}^0_1(\mathcal{S})| > 1$, then $\mathscr{R}^0_1(\mathcal{S})$ is infinite.

Theorem (Selivanov, 1976)

If $|\mathscr{R}^0_1(\mathcal{S})| > 1$, then $\mathscr{R}^0_1(\mathcal{S})$ is not a lattice.

Cardinality and Latticeness of $\mathcal{R}^0_{n+2}(\mathcal{S})$

Theorem (Goncharov, Sorbi, 1997)

1. Let $\mathcal{S}$ be an infinite Σ^0_{n+2}-computable family. Then $\mathcal{R}^0_{n+2}(\mathcal{S})$ contains an infinite subset s.t. any two different elements of the subset form a minimal pair.
2. Let $\mathcal{S}$ be a finite family of Σ^0_{n+2}-sets such that $|\mathcal{S}| > 1$. Then $\mathcal{R}^0_{n+2}(\mathcal{S})$ contains an ideal that is isomorphic to the upper semilattice of c.e. m-degrees $\mathcal{L}^0$.

Corollary (Goncharov, Sorbi, 1997; Ershov, 1969)

Let $\mathcal{S}$ be a Σ^0_{n+2}-computable family such that $|\mathcal{S}| > 1$. Then $\mathcal{R}^0_{n+2}(\mathcal{S})$ is infinite and not a lattice.

Cardinality and Latticeness of $\mathscr{R}_2^h(\mathfrak{F})$

Let $\mathfrak{F}$ be a computable class of families of total functions s.t. $C(\mathfrak{F})$ is finite.

Theorem

Let $\mathcal{S}$ be a finite family of c.e. sets s.t. $\langle \mathcal{S}; \subseteq \rangle \cong \langle C(\mathfrak{F}); \subseteq \rangle$. Then there is an epimorphism from $\mathscr{R}_2^h(\mathfrak{F})$ onto $\mathscr{R}_1^0(\mathcal{S})$.

Corollary

If $C(\mathfrak{F})$ contains two elements which are comparable under inclusion, then $\mathscr{R}_2^h(\mathfrak{F})$ is infinite and not a lattice.

Cardinality and Latticeness of $\mathscr{R}_2^h(\mathfrak{F})$

Let $\mathfrak{F}$ be a computable class of families of total functions s.t. $C(\mathfrak{F})$ is finite and all elements of $C(\mathfrak{F})$ are incomparable under inclusion.

Theorem

If $\mathfrak{F}$ is infinite, then $\mathscr{R}_2^h(\mathfrak{F})$ contains an infinite subset s.t. any two different elements of the subset form a minimal pair.

Theorem

Let $\mathfrak{F}$ be a finite class.

1. If $|\mathfrak{F}| > |C(\mathfrak{F})|$, then $\mathscr{R}_2^h(\mathfrak{F})$ contains an ideal isomorphic to $\mathscr{R}_2^0(\mathscr{A})$ for some finite family of Σ_2^0-sets $\mathscr{A}$ with $|\mathscr{A}| > 1$.
2. If $|\mathfrak{F}| = |C(\mathfrak{F})|$, then $|\mathscr{R}_2^h(\mathfrak{F})| = 1$.

Cardinality and Latticeness of $\mathscr{R}_2^h(\mathfrak{F})$

Corollary

Let $\mathfrak{F}$ be a computable class of families of total functions s.t. $C(\mathfrak{F})$ is finite and $|\mathscr{R}_2^h(\mathfrak{F})| > 1$. Then $\mathscr{R}_2^h(\mathfrak{F})$ is infinite and not a lattice.

Questions

Let $\mathfrak{F}$ be a computable class of families of total functions s.t. $C(\mathfrak{F})$ is infinite.

1. What can we say about the cardinality of $\mathscr{R}_2^h(\mathfrak{F})$?
2. Could the semilattice $\mathscr{R}_2^h(\mathfrak{F})$ be a lattice?

Degree Spectra of the Hereditarily Countable Families

Given a countable structure $\mathfrak{A}$, we define the degree spectrum of $\mathfrak{A}$ to be $\mathrm{Spec}(\mathfrak{A}) = \{X \subseteq \mathbb{N} : \exists \mathfrak{B} \cong \mathfrak{A} \, [\mathfrak{B} \leqslant_T X]\}$.

For an n-family $\mathfrak{F}$, let $\mathrm{Spec}(\mathfrak{F}) = \{X \subseteq \mathbb{N} : \mathfrak{F}$ is X-computable$\}$.

For every n-family $\mathfrak{F}$ there is a structure $\mathfrak{A}(\mathfrak{F})$ s.t. $\mathrm{Spec}(\mathfrak{F}) = \mathrm{Spec}(\mathfrak{A}(\mathfrak{F}))$ (see B. Khoussainov, 1986; C. Ash, J.F. Knight, 2000; S. Goncharov, V. Harizanov, J.F. Knight et al., 2005).

Some examples of degree spectra of families

- **(Slaman; Wehner, 1998)** $\mathrm{Spec}(\mathcal{S}) = \{\mathbf{a} : \mathbf{a} > \mathbf{0}\}$;
- **(Kalimullin, 2008)** $\mathrm{Spec}(\mathcal{S}) = \{\mathbf{a} : \mathbf{a} \nleqslant \mathbf{b}\}$, where $\mathbf{b}$ is c.e.;
- **(Csima, Kalimullin, 2010)** $\mathrm{Spec}(\mathcal{S}) = \{\mathbf{a} : \mathbf{a}$ is h-immune$\}$.

Degree Spectra of the Hereditarily Countable Families

A family $\mathcal{S}$ is finitary if each set $D \in \mathcal{S}$ is finite. An $(n+1)$-family $\mathfrak{F}$ is finitary if each n-family $\mathfrak{D} \in \mathfrak{F}$ is finitary.

Theorem (Slaman; Wehner, 1998)

There is a finitary family $\mathcal{W}$ s.t. $\mathrm{Spec}(\mathcal{W}) = \{\mathbf{a} : \mathbf{a} > \mathbf{0}\}$.

Theorem (Kalimullin, F., 2016)

There is a family $\mathcal{S}$ s.t. $\mathrm{Spec}(\mathcal{S}) = \overline{\mathbf{Low}}_1 = \{\mathbf{a} : \mathbf{a}' > \mathbf{0}'\}$.

Theorem (Kalimullin, F., 2016)

For each $n > 0$ there exist n-families $\mathfrak{F}_n$ and $\mathfrak{S}_n$ s.t. $\mathfrak{F}_n$ is finitary, $\mathrm{Spec}(\mathfrak{F}_n) = \overline{\mathbf{Low}}_{2n-2}$ and $\mathrm{Spec}(\mathfrak{S}_n) = \overline{\mathbf{Low}}_{2n-1}$.

Spectral Hierarchy of the Hereditarily Countable Families

Theorem (Kalimullin, F., 2016)

Let A be a non-low$_n$ c.e. set and $S \in \Sigma^0_{n+3}$. Then there is a computable function f s.t. $\bigoplus_y W_{f(y)} \leqslant_T A$ and $x \in S \Leftrightarrow W_{f(x)}$ is low$_n$, for each x.

Theorem (Kalimullin, F., 2016)

Let $n > 0$. Then for every $(n-1)$-family $\mathfrak{C}$ and every finitary n-family $\mathfrak{D}$, $\mathrm{Spec}(\mathfrak{C}) \neq \overline{\mathbf{Low}}_{2n-2}$ and $\mathrm{Spec}(\mathfrak{D}) \neq \overline{\mathbf{Low}}_{2n-1}$.

Theorem (Goncharov, Harizanov, Knight, McCoy, Miller, Solomon, 2005 – α is a successor; Kalimullin, F., 2016 – α is limit)

For each ordinal $\alpha < \omega_1^{CK}$ there is a structure $\mathfrak{A}$ with $\mathrm{Spec}(\mathfrak{A}) = \overline{\mathbf{Low}}_\alpha = \{\mathbf{a} : \mathbf{a}^{(\alpha)} > \mathbf{0}^{(\alpha)}\}$.

Hereditarily Countable Families of Infinite Rank

Definition

A countable set $\mathfrak{S}$ of hereditarily countable families is said to be a hereditarily countable family of rank α (α-family) if $\alpha = \lim\{\beta : \exists \mathfrak{F} \in \mathfrak{S}\, [\beta \text{ is the rank of } \mathfrak{F}]\}$.

For a computable family $\mathcal{S} = \{W_{f(x)} : x \in \mathbb{N}\}$ each pair $\langle 2, e\rangle$ such that $f = \varphi_e$ is called an enumeration index of $\mathcal{S}$ (note that $|2|_O = 1$).

For $\alpha < \omega_1^{CK}$, a numbering ν of an α-family $\mathfrak{S}$ is computable if there are an $a \in \mathcal{O}$, $|a|_O = \alpha$, and computable functions $f : \mathbb{N} \to \{b \in \mathcal{O} : b <_O a\}$, $g : \mathbb{N} \to \mathbb{N}$ s.t. $\langle f(x), g(x)\rangle$ is an enumeration index of $\nu(x)$ for each x. Let $\varphi_e(x) = \langle f(x), g(x)\rangle$. Then the pair $\langle a, e\rangle$ is called an enumeration index of $\mathfrak{S}$.

Hereditarily Countable Families of Infinite Rank

Let $\mathrm{Spec}(\mathfrak{S}) = \{X \subseteq \mathbb{N} : \mathfrak{S} \text{ is } X\text{-computable}\}$.

Theorem (Kalimullin, F., 2016)

Let α be a countable ordinal. Then for any α-family $\mathfrak{S}$ there is a structure $\mathfrak{A}(\mathfrak{S})$ s.t. $\mathrm{Spec}(\mathfrak{A}(\mathfrak{S})) = \mathrm{Spec}(\mathfrak{S})$.

Theorem (Kalimullin, F., 2016)

Let $\alpha < \omega_1^{CK}$. Then there exists an $(\alpha+1)$-family $\mathfrak{S}$ s.t. $\mathrm{Spec}(\mathfrak{S}) = \overline{\mathbf{Low}}_\alpha$.

Thank you for attention!