Computability and non-computability of planar flows

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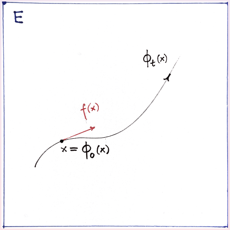
Planar flows: what are they?

A planar flow is the set $\{\phi_t(x) : x \in E \subseteq \mathbb{R}^2, t \in \mathbb{R}\} \subseteq E$ of all solutions to a 2-dimensional differential equation

$$\frac{dx}{dt} = f(x), \quad \text{the vector field } f : E \to \mathbb{R}^2,\ f \in C^1(E)$$

$\phi_t(x)$ is the solution at time t through point x at $t = 0$.

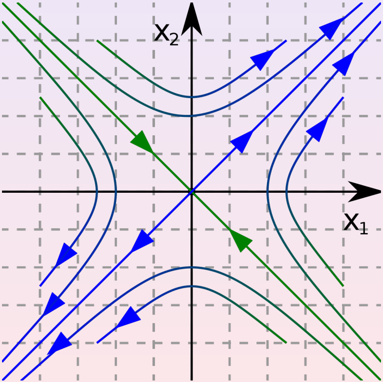
Geometrically, $\phi_t(x)$ is a smooth curve in the phase space E, called a path, a trajectory or an orbit through x.



The phase portrait for the flow = the set of all solution curves

The phase portrait of the coupled linear system

$$\frac{dx_1}{dt} = 3x_1 + x_2, \quad \frac{dx_2}{dt} = x_1 + 3x_2 \quad \text{(the phase space } E = \mathbb{R}^2\text{)}$$



Planar flows: the most wanted?

Main interest: Obtain informative phase portraits.

$\Uparrow$ the idea of Poincaré

Study the qualitative (topological) features of the phase portraits rather than trying to find exact solutions – hopeless for most systems.

Key: Where is the flow approaching to as $t \to \pm\infty$?

$\Uparrow$ called asymptotic states

asymptotic states (singular paths) divide the planar phase portrait into separate regions; each filled with trajectories behaving in the same manner

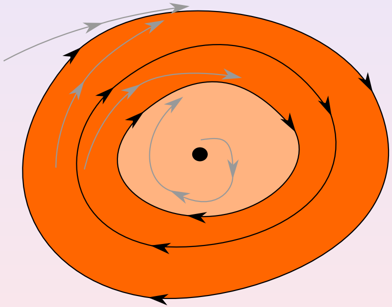
Asymptotic states: from the qualitative viewpoint

Qualitative perspective: relatively simple geometric figures.

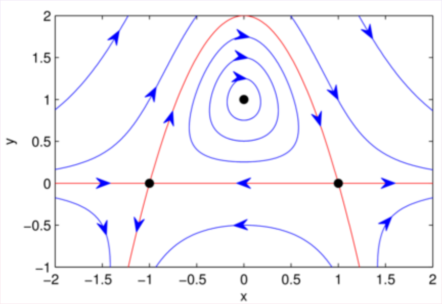
For the **planar system** $dx/dt = f(x)$, only three possible types of asymptotic states (= nonwandering sets):

- equilibrium points ($f(x) = 0 \Rightarrow \phi_t(x) = x$ for all t);
- periodic orbits;
- the unions of saddles and the trajectories connecting them.

Examples of asymptotic states



From center outwards: a source, an attracting, a repelling, and an attracting orbit.



The two saddles are connected.

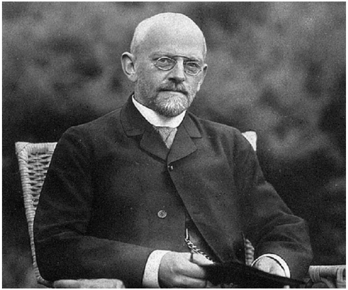
Asymptotic states: from the quantitative viewpoint

Quantitative perspective: many open questions.

The second part of Hilbert's 16th problem: Find the maximum number and relative positions of periodic orbits of the systems of a given degree

$$\frac{dx}{dt} = p(x)$$

the components of $p : \mathbb{R}^2 \to \mathbb{R}^2$ are polynomials of degree n.



Centennial history of Hilbert's 16th problem, Yu. Ilyashenko, Bulletin (new series) of the AMS, Vol 39, No 3 (2002), 301 - 354.

The 2nd part of Hilbert's 16th problem from computability perspective

THE PROBLEM: Study the 2nd part of Hilbert's 16th problem from the computability perspective:

- Can the positions of the periodic orbits be computed for certain classes of polynomials/vector fields on $\mathbb{R}^2$, on a compact subset of $\mathbb{R}^2$, or on a 2D manifold?
- Can other time-invariant sets of a planar flow, such as the equilibrium points, the number of the equilibrium points/periodic orbits (if finite), the basins of attraction, the nonwandering set, be computed?
- Can the computation be uniform on certain classes of polynomials/vector fields?
- What is the computational complexity?

⋮

Results for structurally stable planar systems

Main Result The exact number and positions of the periodic orbits can be computed uniformly on the set of all structurally stable systems defined on a compact disc of $\mathbb{R}^2$.

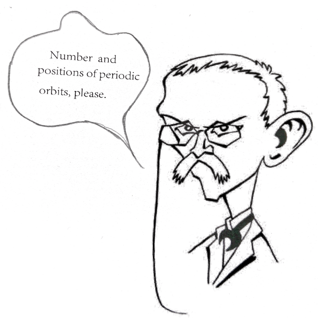
$\Downarrow$

The exact number and positions of the periodic orbits can be uniformly computed on the set of structurally stable polynomial systems on a compact disc of $\mathbb{R}^2$.

Results for structurally stable planar systems

The main result in layman's terms:

I can plot the portrait of your system on my computer screen with whatever precision you wish as long as your system is close-packed and structurally stable.



No problem!
Structurally stable?

Definitions and facts

(1) $\mathcal{X}(\mathbb{D}) = \{f \in C^1(\mathbb{D}) : f$ points inwards along the boundary of $\mathbb{D}\}$, $\mathbb{D} =$ closed unit disc. (Any compact set with a smooth and simple boundary OK.)

(2) $\mathcal{SS}_2(\mathbb{D}) = \{f \in \mathcal{X}(\mathbb{D}) : f$ is <u>structurally stable</u>$\}$

$$\Updownarrow$$

there is $\theta > 0$ such that for any $g \in C^1(\mathbb{D})$ satisfying $\|f - g\|_1 = \max_{x\in\mathbb{D}}\{\|f(x) - g(x)\|, \|Df(x) - Dg(x)\|\} < \theta$, there is a homeomorphism $h : \mathbb{D} \to \mathbb{D}$

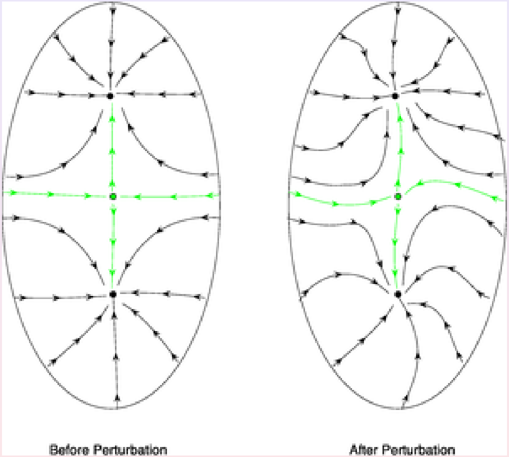
$$\text{trajectories of } \frac{dx}{dt} = f(x) \quad \xrightarrow{h} \quad \text{trajectories of } \frac{dx}{dt} = g(x)$$

$$\Downarrow$$

Small perturbations of f do not alter the topological (qualitative) character of the phase portrait for the flow generated by $dx/dt = f(x)$.

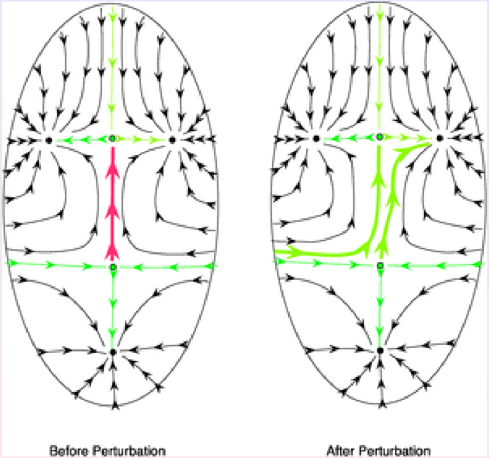
FACT $\mathcal{SS}_2(\mathbb{D})$ is open and dense in $C^1(\mathbb{D}) \Longrightarrow$ Structurally stable systems are typical!

A structurally stable system



limit sets before and after: 2 sinks and 1 saddle

A structurally unstable system



- limit sets before: 3 sinks, 2 saddles, and 1 saddle connection;
- limit sets after: 3 sinks and 2 saddles (saddle connection destroyed)

Main result There is an algorithm that on input (f, k), $f \in \mathcal{SS}_2(\mathbb{D})$ and $k > 0$, outputs the following for the planar flow $dx/dt = f(x)$:

- the exact # of the equilibrium points;
- the squares of side-length $\leq \frac{1}{k}$ each containing one equilibrium and their union contains all equilibrium points;
- the exact # of the periodic orbits;
- the polygonal annuli:
  - ∗ each has the Hausdorff width $\leq \frac{1}{k}$ (Hausdorff width = the Hausdorff distance between the inner and outer boundaries);
  - ∗ each contains at least one periodic orbit; the union contains all periodic orbits.

The preprint is available at http://arxiv.org/abs/2101.07701

Results for structurally stable planar systems

Questions:

- Is the open set $\mathcal{SS}_2(\mathbb{D})$ computable in $C^1(\mathbb{D})$? Or can "$f \in \mathcal{SS}_2(\mathbb{D})$" be decided effectively? ($\mathcal{SS}_2(\mathbb{D})$ is r.e. open.)
- Does the main result remain valid for $\mathcal{SS}_2(\mathbb{R}^2)$, the set of structurally stable systems on $\mathbb{R}^2$?

Results for structurally stable planar systems

Corollary There is an algorithm that on input $f \in \mathcal{SS}_2(\mathbb{D})$ outputs $\{(s, W_s) : s \text{ is a sink of } f\}$, where W_s is the basin of attraction of s:

$$W_s = \{x \in \mathbb{D} : \phi_t(x) \to s \text{ as } t \to \infty\} \quad (\text{open in } \mathbb{R}^2)$$

s is a sink $\iff$ $f(s) = 0$ & trajectories "near" s exponentially converges to s as $t \to \infty$. (W_s is a sinkhole.)

Let W be an open subset of $\mathbb{R}^2$.

- W is r.e. open if it can be filled up by a computable sequence of open pixels.
- W is co-r.e. open if $\mathbb{R}^2 \setminus W$ contains a computable sequence of points that is dense in $\mathbb{R}^2 \setminus W$.
- W is computable if it is r.e. and co-r.e.

About basins of attraction:

- Basins of attraction may have complicated topological structures as subsets of $\mathbb{R}^2$; in fact, many are fractals.
- Basins of attraction vary greatly from system to system.

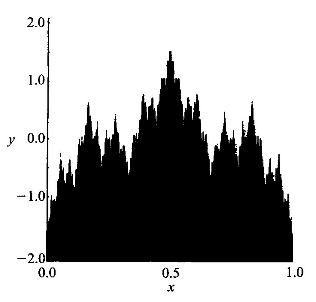
$\Longrightarrow$ Basins of attraction are generally difficult to compute if not impossible.

The basin structure for the map

$$\begin{cases} x_{n+1} = 3x_n \mod 1 \\ y_{n+1} = 1.5y_n + \cos(2\pi x_n) \end{cases}$$

black region = basin of attraction of sink $y = \infty$

blank region = basin of attraction of sink $y = -\infty$



http://www.scholarpedia.org

Structurally unstable planar systems

Many planar polynomial systems are structurally unstable. For example

$$\begin{aligned} \dot{x} &= -\zeta x - \lambda y + xy \\ \dot{y} &= \lambda x - \zeta y + \tfrac{1}{2}(x^2 - y^2) \end{aligned}$$

structurally unstable for $\zeta = 0$ and $\lambda > 0$

Question. What can we say about structurally unstable systems on a compact or on an open set in the plane from computability perspective?

- The structural stability is key to the main result.
- Many structurally unstable systems are computationally bad.

Periodic orbits can be badly non-computable when the system is structurally unstable

Example 1. There is a C^∞ computable function $f : \mathbb{D} \to \mathbb{D}$ such that none of the periodic orbits of the system $dx/dt = f(x)$ is r.e. or co-r.e. as a closed subset of $\mathbb{R}^2$.

Let A be a closed subset of $\mathbb{R}^2$.

- A is co-r.e. if $\mathbb{R}^2 \setminus A$ is r.e. open. ($\rightsquigarrow$ a global property showing an over-adumbration of A after plotting any finite number of pixels).
- A is r.e. if $\mathbb{R}^2 \setminus A$ is co-r.e. open. ($\rightsquigarrow$ a local property - no global picture after plotting finitely many given points).
- A is computable if it is co-r.e. and r.e.

Question: Can the function f be a computable analytic function or a computable polynomial?

The exact number of the periodic orbits may not be uniformly computable on a sequence of polynomials

Example 2. There exists a computable sequence

$$\mathcal{P} = \{p_k\}, \quad p_k \text{ is a 3rd degree planar polynomial}$$

such that the map $\Phi : \mathcal{P} \to \mathbb{N}$, $\Phi(p_k) =$ the number of periodic orbits of the system $dx/dt = p_k(x)$, is continuous but $\underbrace{\text{non-computable}}$.

$\Updownarrow$

the halting problem

Basins of attraction can be persistently non-computable in $\mathbb{R}^2$

Example 3. Let $K = \{x \in \mathbb{R}^2 : \|x\| \leq 3\}$.

(1) There exists a computable C^∞ function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f \in \mathcal{X}(K)$, the system $dx/dt = f(x)$ has a unique computable sink whose basin of attraction W_f is non-computable.

(2) For any C^1-neighborhood U of f, there exists a computable C^∞ function g in $\mathcal{X}(K)$ such that $g \in U$, $g \neq f$, and the system $dx/dt = g(x)$ has a unique computable sink whose basin of attraction W_g is non-computable.

Non-computability can be (non-trivially) persistent under perturbations.

Basins of attraction of an analytic system can be robustly non-computable in $\mathbb{R}^3$

Example 4.

(1) There exists a computable analytic function $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that the discrete system generated by f has a computable sink but its basin of attraction is non-computable.

(2) There is a C^1-neighborhood $\mathcal{N}$ of f (computable from f and $Df(s)$) such that for each and every $g \in \mathcal{N}$, g has a sink (computable from g) whose basin of attraction is non-computable.

Non-computability can be pervasive in an entire neighborhood of f: every function in this neighborhood has a non-computable basin of attraction.

Proof outline of the main result

The proof consists of 3 algorithms (A), (B), and (C):

(A) computes the number of and locates the positions of the equilibrium points;

(B) locates the positions of the periodic orbits; and

(C) computes the number of the periodic orbits.

Proof outline of the main result: algorithm (A)

Math underlying (A): Let $dx/dt = f(x)$, $f \in \mathcal{SS}_2(\mathbb{D})$.

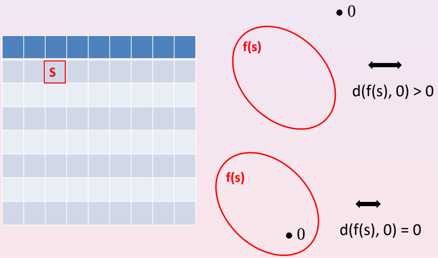
- It has only finitely many equilibrium points $\Longrightarrow$ possible to compute the exact number of them.
- Each equilibrium x is hyperbolic $\Longrightarrow$ $f(x) = 0$ and $Df(x)$ is invertible $\Longrightarrow$ possible to construct (A) using a computable version of the inverse function theorem.
- Hyperbolic equilibria are robust under small perturbations on f $\Longrightarrow$ possible to use a name of f as an input (a name of f = a poly sequence approximating f in C^1-norm).

The facts also hold true for periodic orbits.

Proof outline of the main result: (A) continued

INPUT: $n_0 \geq 1$ (accuracy) and a C^1-name of f; OUTPUT: a set of squares with side length $\leq 1/n_0$ each contains exactly one equilibrium.

Cover $\mathbb{D}$ with a rational square-grid: s has side-length $1/n$, $n > 3n_0$.



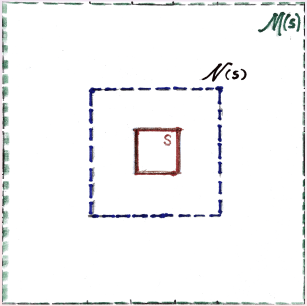
$d(f(s), 0) > 0$ computable; $d(f(s), 0) = 0$ non-computable

Proof outline of the main result: (A) continued

(1) Compute $d(f(s), 0)$ and $\min\{\|Df(x)\|, |\det Df(x)| : x \in \mathcal{M}(s)\}$ (increasing n if necessary) until

$$\underline{d(f(s), 0) > 2^{-n}} \quad \text{or} \quad \min\{\|Df(x)\|, |\det Df(x)| : x \in \mathcal{M}(s)\} > 2^{-n}$$

$\Uparrow$ discard s (s contains no equilibrium)



Proof outline of the main result: (A) continued

(2) Assume that $\min\{\|Df(x)\|, |\det Df(x)| : x \in \mathcal{M}(s)\} \geq 2^{-n}$. **Idea:** Refine $s = \bigcup s_j$ so that

$$f(s_j) \subset f(\bigcup B(x_i, \alpha_i)) \subseteq \bigcup_{i=1}^{J} \underline{B(f(x_i), \beta_i)} \subset f(\mathcal{N}(s_j))$$

$$\Uparrow$$

"$0 \in B(f(x_i), \beta_i)$?" decidable effectively

(3) Output $\underline{\mathcal{N}(s_j)}$ if $0 \in \bigcup B(f(x_i), \beta_i)$; discard s_j otherwise.

$$\Updownarrow$$

$\mathcal{N}(s_j)$ contains a unique equilibrium with side length $\leq \frac{3}{n} \leq \frac{1}{n_0}$

Proof outline of the main result: algorithm (B)

(B) is sophisticated: Time comes into play!

To detect periodic orbits $\overset{\text{need}}{\Longleftarrow}$ find where the trajectories are approaching to as $t \to \pm\infty$ on the entire $\mathbb{D}$ $\overset{\text{possible}}{\Longleftarrow}$ compute the motion of the flow for more and more points in $\mathbb{D}$ over longer and longer time periods

$\Uparrow$ To be able to halt the computation

Need a uniform time bound by which time (forward and backward) all trajectories starting at sample points would have already gathered around all asymptotic states (to be found).

Recall that periodic orbits are asymptotic states; for every point $p \in \mathbb{D}$ the trajectory starting at p will converge to an asymptotic state as $t \to \infty$ or $t \to -\infty$.

Proof outline of the main result: (B) continued

Old and new tools from numerical analysis, dynamical systems and computable analysis used to construct (B):

- Peixoto's characterization theorem: A structurally stable planar system has only finitely many equilibria/periodic orbits as its asymptotic states; all hyperbolic.
- The Poincaré-Bendixson theorem $\Longrightarrow$ An time-invariant compact region in the phase space containing no equilibria must contain a periodic orbit.
- Persistence of hyperbolic equilibrium points and periodic orbits.
- A rigorous numerical method for computing (flow) images of lattices.
- A computable version of the stable manifold theorem.
- A computable version of the Hartman-Grobman Theorem.
- A coloring program for identifying "donut" shaped regions in the phase space.

⋮

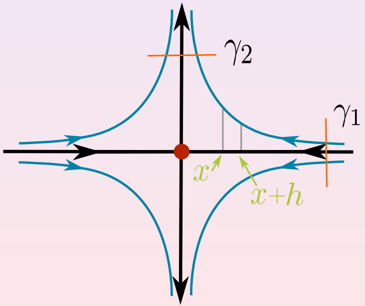
Proof outline of the main result: (B) continued

Special case: the system has no saddles

Why are saddles troublesome? The uniform time bound is in jeopardy!

$\Uparrow$ Why?

It may take arbitrarily long time for points near a saddle moving away from it and reaching the neighborhoods of some other asymptotic states.



The closer a point to the red without on x-axis, the longer it takes to reach γ_2

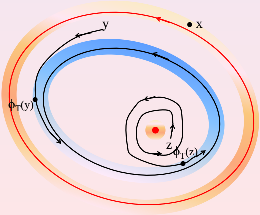
Proof outline of the main result: (B) continued

What's good if no saddles: For any $\theta > 0$, there is a time $T_\theta > 0$ s.t.

$$\forall p \in \mathbb{D} \begin{cases} p \text{ already } \theta\text{-close to a repeller} \\ \phi_t(p) \text{ in } \theta\text{-neighborhood of some attractor for all } t \geq T_\theta \end{cases}$$

$\Longrightarrow$ T_θ is a (theoretical) time bound – forward or backward – for all points moving into the θ-neighborhood of the asymptotic states (via trajectories).

$\Longrightarrow$ Possible to "catch" all asymptotic states within distance θ by time T_θ by following "sufficiently many" points!



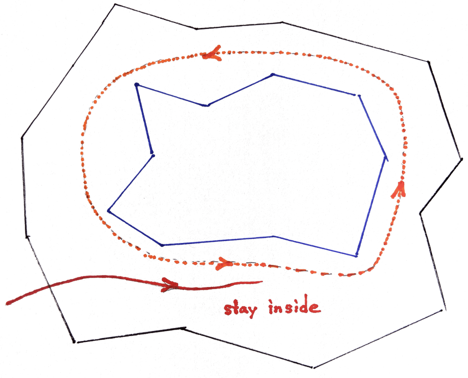
x is θ-close to a repeller; $\phi_T(y)$ and $\phi_T(z)$ enter the θ-neighborhood of an attractor and stay there happily thereafter.

Proof outline of the main result: (B) continued

Algorithm (B) in the spacial case:

Wanted: Locate all periodic orbits. Idea: Donut hunt.

Find donut-shaped flow images in $\mathbb{D}$ for sufficiently many sample points over sufficiently long time periods $\overset{\text{why}}{\Longleftarrow}$ If a donut contains no equilibrium and keeps all trajectories inside it from leaving, then it contains at least one periodic orbit by Poincarè-Bendixson theorem.



Proof of the main result – (B) continued – donut hunt

A road map for hunting good skinny donuts (a flow-chart for (B)):

- Cover $\mathbb{D}$ with finitely many square pixels, simulate the flow using a rigorous numerical method, and compute the (simulated flow) images of pixels for some integer time T (and $-T$ simultaneously).

? Are the images time-invariant from now on?

? If yes, are the time-invariant connected components donut shaped?

? If yes, is each donut good and containing no equilibrium?

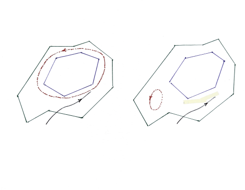
? If yes, are the donuts mutually disjoint?

? If yes, is each donut skinny enough?

- If yes, the happy end.
- Whenever encountering NO, restart the journey with an increased time and decreased pixel size.

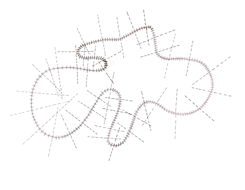
Proof outline of the main result: (B) continued – donut hunt

The good donuts can be detected by a coloring program.

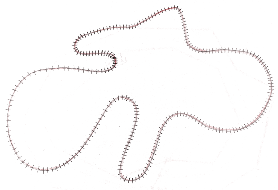


The left is good; the right is bad because the donut is not a good approximation of the periodic orbit.

A “haircut” theorem is established for halting the coloring program: For a C^2 simple closed curve, there is $\delta > 0$ such that the hairs – growing in the normal direction – can be cut uniformly with length δ and the tips of hairs do not tangle after the cut.



Before the cut during pandemic lockdown



Have a cut after vaccination

Proof outline of the main result: (B) continued

Full picture: Saddles exist in the system.

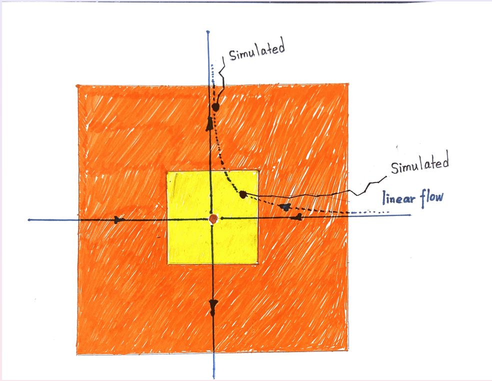
Potential trouble with saddles – no time bound for points near a saddle to move away from it.

Method developed to tackle the problem:

- use (A) to identify the saddles;
- at each saddle, use a computable version of Hartman-Grobman's theorem to identify a small neighborhood V and then transform the origin flow in V to a linear flow;
- use the linear system on V – can be computed explicitly – as an oracle to supply a good exit-approximation to every simulated trajectory entering V, in one unit of time.

$\Longrightarrow$ Uniform time bounds preserved.

Proof outline of the main result: (B) continued



The linear dynamics on the orange box acting as an oracle to the original flow.

A simulated trajectory enters the yellow square $\Longrightarrow$ the linear system picks up a point on it, computes the linear flow starting at this point until it reaches the orange region with sufficiently good accuracy $\Longrightarrow$ the original system picks up a point on the linear trajectory in the orange region and resumes its activity.

Proof outline of the main result: algorithm (C)

INPUT: the mutually disjoint good skinny donuts (= the output of (B))

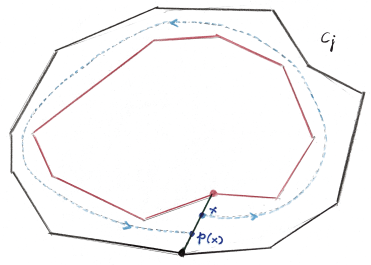
OUTPUT: the number of periodic orbits

(1) For each C_j (a donut with polygonal interior- and exterior-boundary), use a line segment l_j from one vertex on the interior boundary to the nearest vertex on the exterior boundary as a cross-section of C_j.

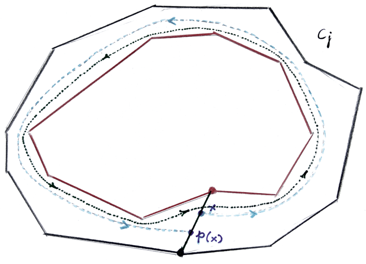
(2) Show that the Poincaré map P_j on l_j and its derivative are computable.

(3) the number of periodic orbits inside C_j = the number of fixed points of P_j; the latter can be computed by algorithm (A).

Proof outline of the main result: (C) continued



The first return map.



A fixed point of the first return map corresponds to a periodic orbit.

Thank you