On Guarded Extensions of MMSNP

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lacl

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Fagin's theorem (1974)

The set of problems expressible with existential second-order logic is exactly the class NP.

Ladner's theorem (1975)

NO dichotomy

If P≠NP, then there are problems in NP that are neither in P nor NP-complete.

Monotone Monadic SNP without Inequality

Definition ([Feder, Vardi, 1998])

The MMSNP logic consists of ESO sentences of the form

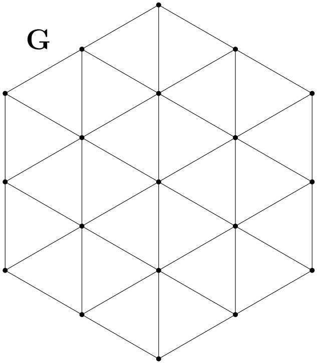
$$\exists X_1, \ldots, X_s \, \forall x_1, \ldots, x_n \bigwedge_{i=1}^{m} \neg\big(\alpha_i \wedge \beta_i \wedge \varepsilon_i\big), \text{ where}$$

- every α_i is a conjunction of input atomic formulas,
- every β_i is a conjunction of existential atomic formulas,
- every ε_i is a conjunction of inequalities $(x_j \neq x_k)$,
- all atomic formulas of α_i must be non-negated (monotone),
- all existential relations $X_1, \ldots, X_s$ have arity 1 (monadic),
- every ε_i is empty (without inequality).

Example

No Monochromatic Triangle

Given a graph $\mathbf{G}$, color its vertices with 2 colors so that the result omits the two following subgraphs.



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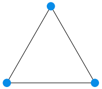
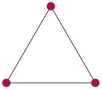
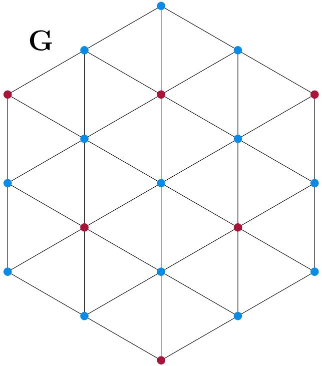
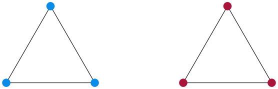






"No Monochromatic Triangle" as a sentence in MMSNP



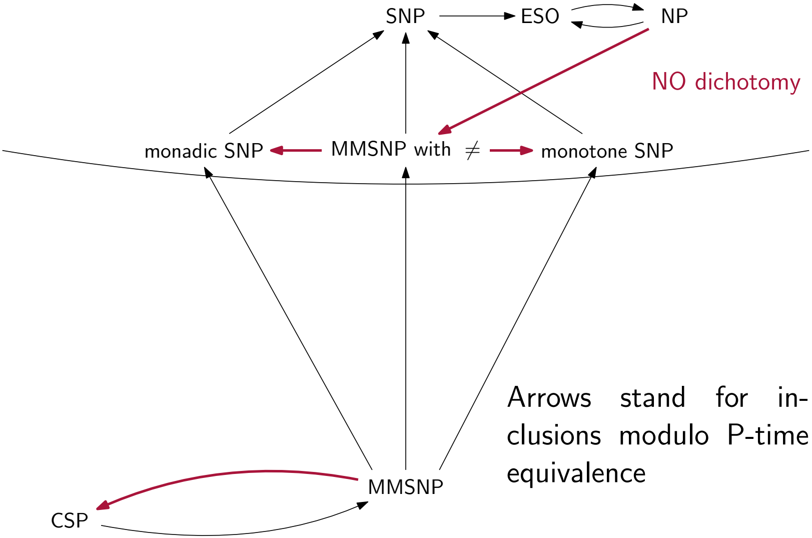
$\exists R, B\ \forall x, y, z$

$\neg(\neg Rx \wedge \neg Bx) \wedge \neg(Rx \wedge Bx)$ (R and B partition the elements)

$\wedge\neg(Exy \wedge Eyz \wedge Ezx \wedge Rx \wedge Ry \wedge Rz)$ (no all-red triangle)

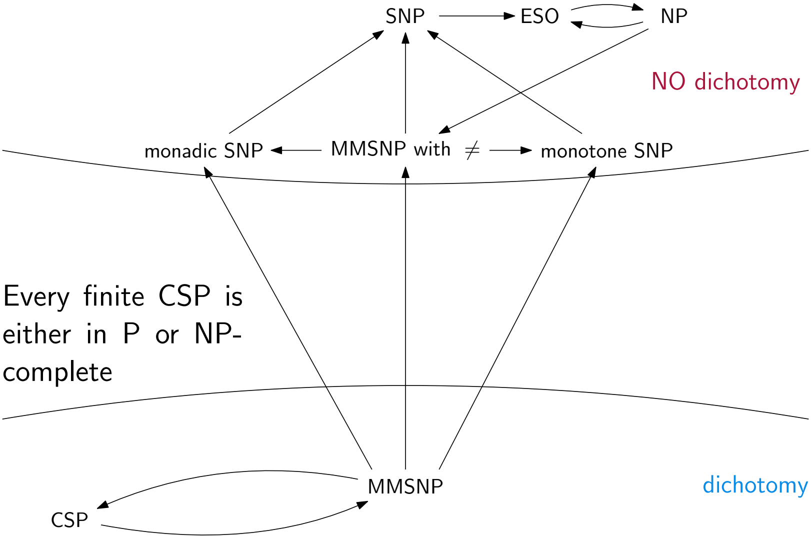
$\wedge\neg(Exy \wedge Eyz \wedge Ezx \wedge Bx \wedge By \wedge Bz)$ (no all-blue triangle)

Feder and Vardi's results (1998)



Arrows stand for inclusions modulo P-time equivalence

Zhuk's theorem (2017)



Every finite CSP is either in P or NP-complete

Guarded Monotone SNP without Inequality

Definition ([Bienvenu et al., 2014])

The GMSNP logic consists of ESO sentences of the form

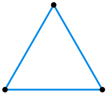
$$\exists X_1, \ldots, X_s \, \forall x_1, \ldots, x_n \bigwedge_{i=1}^{m} \neg(\alpha_i \wedge \beta_i), \text{ where}$$

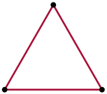
- every α_i is a conjunction of input atomic formulas,
- every β_i is a conjunction of existential atomic formulas,
- all atomic formulas of α_i must be non-negated (monotone),
- for every $X_j(\mathbf{t})$ in β_i there exists $R(\mathbf{u})$ in α_i such that $\mathbf{t} \subseteq \mathbf{u}$ (i.e., $\mathbf{t}$ is guarded by $\mathbf{u}$).

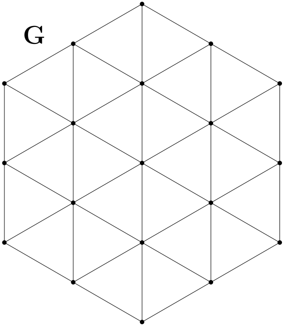
Example

No Monochromatic Edge Triangle

Given a graph $\mathbf{G}$, color its edges with 2 colors so that the result omits the two following subgraphs.







Example

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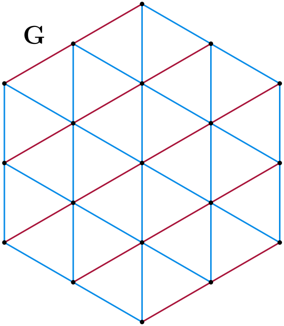
Given a graph $\mathbf{G}$, color its edges with 2 colors so that the result omits the two following subgraphs.



“No Monochromatic Edge Triangle” as a GMSNP sentence

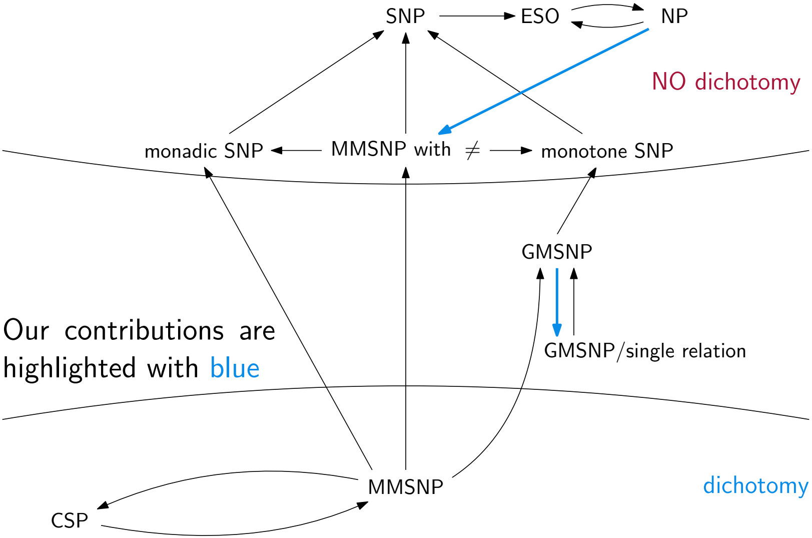
$\exists R, B\ \forall x, y, z$

$\neg(Exy \wedge \neg Rxy \wedge \neg Bxy) \wedge \neg(Exy \wedge Rxy \wedge Bxy)$ (partition of edges)

$\wedge\neg(Exy \wedge Eyz \wedge Ezx \wedge Rxy \wedge Ryz \wedge Rzx)$ (no all-red triangle)

$\wedge\neg(Exy \wedge Eyz \wedge Ezx \wedge Bxy \wedge Byz \wedge Bzx)$ (no all-blue triangle)

Guarded Monotone SNP without Inequality (2014)



Our contributions are highlighted with blue

Monotone Monadic SNP with Guarded Inequality

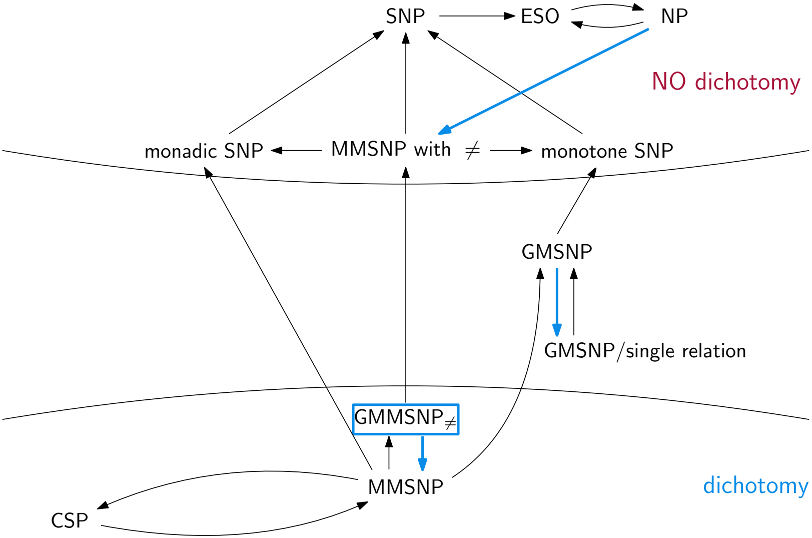
Definition (B., Madelaine)

The GMMSNP$_{\neq}$ logic consists of ESO sentences of the form

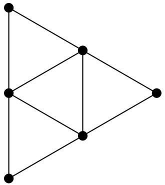
$$\exists X_1, \dots, X_s \, \forall x_1, \dots, x_n \bigwedge_{i=1}^{m} \neg\big(\alpha_i \wedge \beta_i \wedge \varepsilon_i\big), \text{ where}$$

- every α_i is a conjunction of input atomic formulas,
- every β_i is a conjunction of existential atomic formulas,
- every ε_i is a conjunction of inequalities $(x_j \neq x_k)$,
- all atomic formulas of α_i must be non-negated (monotone),
- all existential relations $X_1, \dots, X_s$ have arity 1 (monadic),
- for every $x_j \neq x_k$ in ε_i there exists $R(\mathbf{u})$ in α_i such that $x_j, x_k \in \mathbf{u}$ (guarded inequality).

Monotone Monadic SNP with Guarded Inequality



Matrix Partition



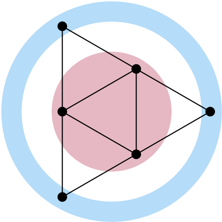
$$M = \begin{bmatrix} 0 & * \\ * & 1 \end{bmatrix}$$

Definition (Feder, Hell)

Let M be a square matrix of size m with elements from $\{0, 1, *\}$. Given an input digraph, split its vertices into disjoint classes $P_1, \ldots, P_m$ such that, for $i, j \in [m]$ and any distinct $x \in P_i, y \in P_j$:

- if $M(i, j) = 0$, then there is no edge between x and y;
- if $M(i, j) = 1$, then there is an edge between x and y;
- if $M(i, j) = *$, then there is no restriction.

Matrix Partition



$$M = \begin{bmatrix} 0 & \ast \\ \ast & 1 \end{bmatrix}$$

Definition (Feder, Hell)

Let M be a square matrix of size m with elements from $\{0, 1, \ast\}$. Given an input digraph, split its vertices into disjoint classes $P_1, \ldots, P_m$ such that, for $i, j \in [m]$ and any distinct $x \in P_i, y \in P_j$:

- if $M(i, j) = 0$, then there is no edge between x and y;
- if $M(i, j) = 1$, then there is an edge between x and y;
- if $M(i, j) = \ast$, then there is no restriction.

Towards a Logic for Matrix Partition

Definition (B., Madelaine)

The MPART logic consists of ESO sentences of the form

$$\exists X_1, \ldots, X_s \, \forall x_1, \ldots, x_n \bigwedge_{i=1}^{m} \neg\big(\alpha_i \wedge \beta_i\big), \text{ where}$$

- every α_i is a conjunction of input atomic formulas,
- every β_i is a conjunction of existential atomic formulas,
- all existential relations $X_1, \ldots, X_s$ have arity 1 (monadic),
- atomic formulas in α_i are either all non-negated or all negated (same polarity).

MPART with Guarded Inequality

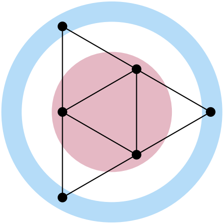
Definition (B., Madelaine)

The GMPART$_{\neq}$ logic consists of ESO sentences of the form

$$\exists X_1, \ldots, X_s \, \forall x_1, \ldots, x_n \bigwedge_{i=1}^{m} \neg\left(\alpha_i \wedge \beta_i \wedge \varepsilon_i\right), \text{ where}$$

- $\alpha_i, \beta_i, \varepsilon_i$ are the same as before,
- all existential relations $X_1, \ldots, X_s$ have arity 1 (monadic),
- atomic formulas in α_i are either all non-negated or all negated (same polarity),
- for every $x_j \neq x_k$ in ε_i there exists $R(\mathbf{u})$ in α_i such that $x_j, x_k \in \mathbf{u}$ (guarded inequality).

Split graph recognition as a GMPART$_{\neq}$ sentence



$$M = \begin{bmatrix} 0 & * \\ * & 1 \end{bmatrix}$$

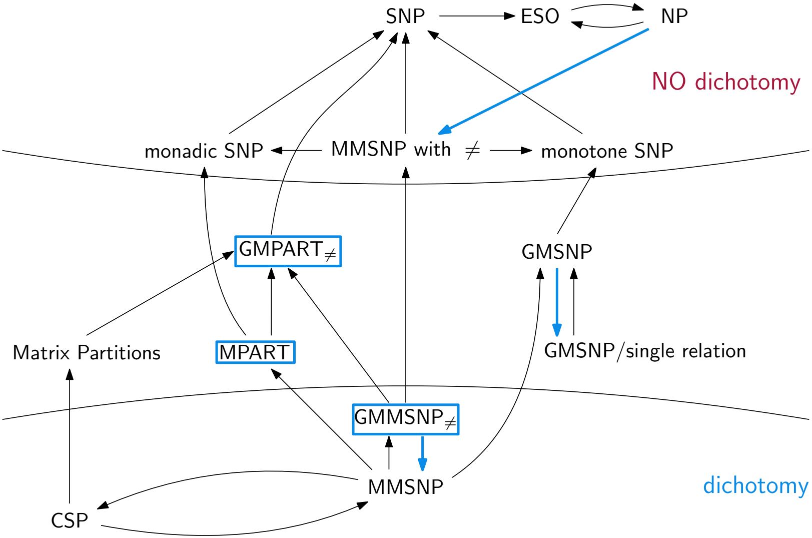
$\exists C, I\ \forall x, y$

$\neg(Cx \land \neg Ix) \land \neg(Cx \land Ix)$ (partition of vertices in 2 classes)

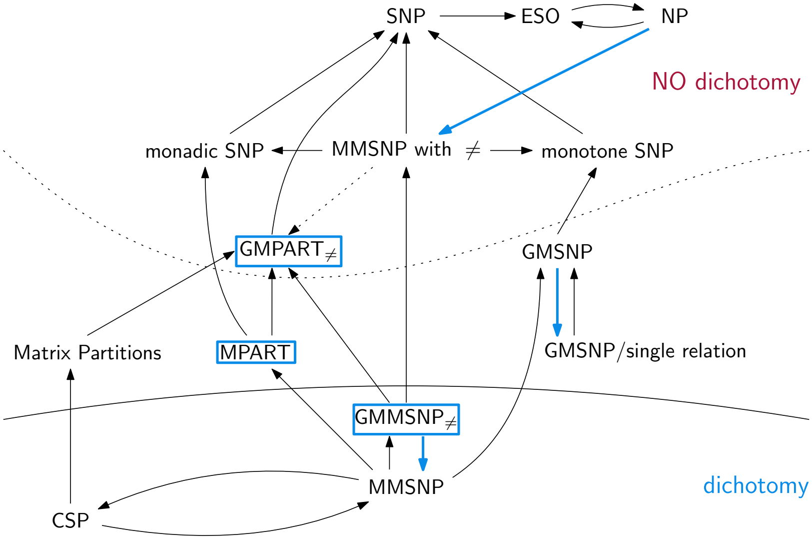
$\land \neg(\neg Exy \land Cx \land Cy \land x \neq y)$ (C induces a clique)

$\land \neg(Exy \land Ix \land Iy \land x \neq y)$ (I induces an independent set)

Matrix Partition and MPART and GMPART$_{\neq}$



GMPART$_{\neq}$ has NO dichotomy



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