A Constructive Picture of Noetherianity and Well Quasi-Orders

Gabriele Buriola[1], Peter Schuster[1], Ingo Blechschmidt[2]

[1]University of Verona, Italy
[2]University of Augsburg, Germany

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Summary

We will see:

- Constructive Noetherian definitions;
- Constructive well quasi-orders and their relations.

Ascending chain condition, classically

Classical logic := Excluded Middle (LEM) + Axiom of Choice (AC).

Classical Noetherianity for Rings

- FBP (Finite Basis Property): every ideal is finitely generated;
- ACC: every ascending chain of ideals *stabilizes*

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots ;$$

- Classically: FBP $\leftrightarrow$ ACC.

Problem:

FBP and ACC are not constructively meaningful!

E.g. the 2 element field $\mathbb{F}_2$ is neither constructively FBP nor ACC.

Ascending chain condition, constructively

Toward a constructive ACC

- ACC: every ascending chain of ideals stabilizes
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots ;$$
- ACC^{fg}: every ascending chain of finitely generated ideals stabilizes
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1} = I_{n+2} = \dots ;$$
- ACC_0: every ascending chain of ideals *stalls*
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$
- ACC_0^{fg}: every ascending chain of finitely generated ideals stalls
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \quad \Rightarrow \quad \exists n : I_n = I_{n+1};$$

ACC: is *not* constructive, e.g. $\mathbb{F}_2$;

ACC^{fg}: is *not* constructive, e.g. by Halting problem for Turing machines;

ACC_0: is *not* constructive, e.g. by topological models of intuitionistic logic;

ACC_0^{fg}: is constructive! as discovered by Richman and Seidenberg;

Notation: RS-Noetherian:=ACC_0^{fg} (d'après Richman and Seidenberg).

Related Properties

Let $(E, \leqslant)$ be a partial order with $x < y \equiv x \leqslant y \wedge x \neq y$:

Hereditary conditions

- $H \subseteq E$ is hereditary if $\forall x(\{y \mid y < x\} \subseteq H \Rightarrow x \in H)$;
- E is hereditary well-founded, hwf, if $H \subseteq E$ hereditary $\Rightarrow H = E$;
- E is well ordered if it is hereditary well-founded and linear.

Ascending trees (Richman'03)

An ascending tree in E is a family $(x_i)_{i \in I} \subseteq E$ where

- I is a tree;
- $i < j \Rightarrow x_i \leqslant x_j$.

An ascending tree *stalls* if $\exists\, i < j : x_i = x_j$.

Inductive definition of "P bars σ"

For a predicate P on ascending finite lists on E, we define $P|\sigma$:

- if $P(\sigma)$ then $P|\sigma$;
- if $P|\sigma x$ for all $x \geqslant \sigma$, then $P|\sigma$.

Intuitionistic Noetherian properties and their relations

A partial order $(E, \leqslant)$ is

- RS-Noetherian if for $e_1 \leqslant e_2 \leqslant \ldots$ there is n with $e_n = e_{n+1}$;
- ML-Noetherian if the reverse order $(E, \geqslant)$ is hwf;
- strongly Noetherian if there is a well-order W and a strictly descending map $\varphi\colon E \to W$, i.e. $e < f \Rightarrow \varphi(e) > \varphi(f)$;
- tree Noetherian if every ascending tree in E stalls;
- inductively Noetherian if $\mathrm{Stall} \mid []$, where $\mathrm{Stall}(\sigma)=$"σ is an ascending finite list with repeated terms".

Def: given a ring R, $\mathcal{I}_f(R)$ is the set of finitely generated ideals of R.
Def: a ring R is * Noetherian if $(\mathcal{I}_f(R), \subseteq)$ is * Noetherian.

Constructive implications for a decidable poset $(E, \leqslant)$

$$\text{strong} \longrightarrow \text{ind} \longrightarrow \text{tree} \longrightarrow \text{RS}$$
$$\text{ind} \longleftrightarrow \text{ML}$$

Basic definitions for quasi-orders

Quasi-order

A qo $(Q, \leqslant)$ is a set Q with a transitive and reflexive relation $\leqslant$.

Notation

- $p < q \;\equiv\; p \leqslant q \;\wedge\; q \nleqslant p$;
- $p \perp q \;\equiv\; p \nleqslant q \;\wedge\; q \nleqslant p$;
- $p \sim q \;\equiv\; p \leqslant q \;\wedge\; q \leqslant p$.

Auxiliary definitions

For every qo $(Q, \leqslant)$:

- the closure of $B \subseteq Q$ is $\uparrow B := \{q \in Q \mid \exists b \in B \; b \leqslant q\}$;
- B is closed if $B{=}\uparrow B$ and finitely generated if $B{=}\uparrow\{b_1, \ldots, b_n\}$;
- a sequence $(q_k)_k$ in Q is a total function from $\mathbb{N}$ to Q;
- an antichain is a sequence $(q_k)_k$ such that $q_i \perp q_j$ if $i \neq j$;
- an extension of $(Q, \leqslant)$ is a qo $\preccurlyeq$ on Q extending $\leqslant$, i.e., $p \leqslant q \Rightarrow p \preccurlyeq q$ and $p \preccurlyeq q \wedge q \preccurlyeq p \Rightarrow p \sim q$.

Well quasi-orders definitions

A qo $(Q, \leqslant)$ is

- well-founded if for $q_1 \geqslant q_2 \geqslant \dots$ there is n such that $q_n = q_{n+1}$;
- wqo if for any sequence $(q_k)_k$ in Q there exist $i < j$ with $q_i \leqslant q_j$;
- wqo(set) if every sequence $(q_k)_k$ in Q has an infinite ascending subsequence: there are $k_0 < k_1 < \dots$ such that $q_{k_0} \leqslant q_{k_1} \leqslant \dots$;
- wqo(anti) if it is well-founded and every antichain is finite;
- wqo(ext) if every linear extension of Q is well-founded;
- wqo(fbp) if every closed subset is finitely generated;
- wqo(acc) if the set of closed subsets is Noetherian;
- wqo(*) if the set of finitely generated closed subsets is *Noetherian.

Remark: all the wqo definitions are classically equivalent.

Constructive relations between wqo definitions

Theorem

The conditions wqo(set), wqo(fbp) and wqo(acc) are not constructively meaningful.

Implications between constructive wqo definitions

wqo(str) ⟶ wqo(ML) ⟶ wqo(RS) ⟶ wqo

wqo ↗ wqo(anti)

wqo ↘ wqo(ext)

A closure property

Let $\mathcal{P}$ any of the properties wqo, wqo(anti), . . . except wqo(ext).
If $(Q, \leqslant)$ has property $\mathcal{P}$ and $P \subseteq Q$, then $(P, \leqslant)$ has property $\mathcal{P}$.

Future work

Well-founded vs. hereditarily well-founded

Classically equivalent, but not constructively.

Reverse implications

Which of the following implications can be reversed?

- strongly Noetherian $\Rightarrow$ ML-Noetherian;
- wqo(RS) $\Rightarrow$ wqo;
- wqo $\Rightarrow$ wqo(anti);
- ...

For now, RS-Noetherian $\not\Rightarrow$ ML-Noetherian by A. Blass.

Further closure properties

Is wqo(ext) closed under subset?

If P and Q have property $\mathcal{P}$, does

- $P \dot{\cup} Q$ constructively have property $\mathcal{P}$?
- $P \times Q$ constructively have property $\mathcal{P}$?

Thank you! References:

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