

A Constructive Picture of Noetherianity and Well Quasi-Orders

Gabriele Buriola₁, Peter Schuster₁, Ingo Blechschmidt₂

¹University of Verona, Italy

²University of Augsburg, Germany

28/07/2023

Summary

We will see:

- Constructive Noetherian definitions;
- Constructive well quasi-orders and their relations.

Ascending chain condition, classically

Classical logic := Excluded Middle (LEM) + Axiom of Choice (AC).

Classical Noetherianity for Rings

- FBP (Finite Basis Property): every ideal is **finitely generated**;
- ACC: every ascending chain of ideals **stabilizes**

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \Rightarrow \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$

- **Classically**: FBP \leftrightarrow ACC.

Problem:

FBP and ACC are not constructively meaningful!

E.g. the 2 element field \mathbb{F}_2 is neither constructively FBP nor ACC.

Ascending chain condition, constructively

Toward a constructive ACC

- ACC: every ascending chain of ideals **stabilizes**

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \Rightarrow \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$

- ACC^{fg} : every ascending chain of **finitely generated** ideals stabilizes

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \Rightarrow \exists n : I_n = I_{n+1} = I_{n+2} = \dots;$$

- ACC_0 : every ascending chain of ideals **stalls**

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \Rightarrow \exists n : I_n = I_{n+1};$$

- ACC_0^{fg} : every ascending chain of **finitely generated** ideals **stalls**

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots \Rightarrow \exists n : I_n = I_{n+1};$$

ACC: is *not* constructive, e.g. \mathbb{F}_2 ;

ACC^{fg} : is *not* constructive, e.g. by Halting problem for Turing machines;

ACC_0 : is *not* constructive, e.g. by topological models of intuitionistic logic;

ACC_0^{fg} : is **constructive!** as discovered by Richman and Seidenberg;

Notation: RS-Noetherian := ACC_0^{fg} (d'après Richman and Seidenberg).

Related Properties

Let (E, \leq) be a partial order with $x < y \equiv x \leq y \wedge x \neq y$:

Hereditary conditions

- $H \subseteq E$ is **hereditary** if $\forall x(\{y \mid y < x\} \subseteq H \Rightarrow x \in H)$;
- E is **hereditary well-founded**, **hwf**, if $H \subseteq E$ hereditary $\Rightarrow H = E$;
- E is **well ordered** if it is hereditary well-founded and linear.

Ascending trees (Richman'03)

An **ascending tree** in E is a family $(x_i)_{i \in I} \subseteq E$ where

- I is a tree;
- $i < j \Rightarrow x_i \leq x_j$.

An ascending tree **stalls** if $\exists i < j : x_i = x_j$.

Inductive definition of "P bars σ "

For a predicate P on ascending finite lists on E , we define $P|\sigma$:

- if $P(\sigma)$ then $P|\sigma$;
- if $P|\sigma x$ for all $x \geq \sigma$, then $P|\sigma$.

Intuitionistic Noetherian properties and their relations

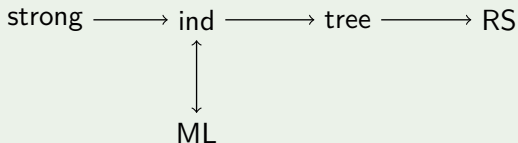
A partial order (E, \leq) is

- **RS-Noetherian** if for $e_1 \leq e_2 \leq \dots$ there is n with $e_n = e_{n+1}$;
- **ML-Noetherian** if the reverse order (E, \geq) is hwf;
- **strongly Noetherian** if there is a well-order W and a strictly descending map $\varphi: E \rightarrow W$, i.e. $e < f \Rightarrow \varphi(e) > \varphi(f)$;
- **tree Noetherian** if every ascending tree in E stalls;
- **inductively Noetherian** if $\text{Stall} \mid []$, where $\text{Stall}(\sigma) = \text{“}\sigma \text{ is an ascending finite list with repeated terms”}$.

Def: given a ring R , $\mathcal{I}_f(R)$ is the set of finitely generated ideals of R .

Def: a ring R is $*$ Noetherian if $(\mathcal{I}_f(R), \subseteq)$ is $*$ Noetherian.

Constructive implications for a decidable poset (E, \leq)



Basic definitions for quasi-orders

Quasi-order

A **qo** (Q, \leq) is a set Q with a **transitive** and **reflexive** relation \leq .

Notation

- $p < q \equiv p \leq q \wedge q \not\leq p$;
- $p \perp q \equiv p \not\leq q \wedge q \not\leq p$;
- $p \sim q \equiv p \leq q \wedge q \leq p$.

Auxiliary definitions

For every qo (Q, \leq) :

- the **closure** of $B \subseteq Q$ is $\uparrow B := \{q \in Q \mid \exists b \in B \ b \leq q\}$;
- B is **closed** if $B = \uparrow B$ and **finitely generated** if $B = \uparrow \{b_1, \dots, b_n\}$;
- a **sequence** $(q_k)_k$ in Q is a total function from \mathbb{N} to Q ;
- an **antichain** is a sequence $(q_k)_k$ such that $q_i \perp q_j$ if $i \neq j$;
- an **extension** of (Q, \leq) is a qo \preceq on Q extending \leq , i.e.,
 $p \leq q \Rightarrow p \preceq q$ and $p \preceq q \wedge q \preceq p \Rightarrow p \sim q$.

Well quasi-orders definitions

A qo (Q, \leq) is

- **well-founded** if for $q_1 \geq q_2 \geq \dots$ there is n such that $q_n = q_{n+1}$;
- **wqo** if for any sequence $(q_k)_k$ in Q there exist $i < j$ with $q_i \leq q_j$;
- **wqo(set)** if every sequence $(q_k)_k$ in Q has an infinite ascending subsequence: there are $k_0 < k_1 < \dots$ such that $q_{k_0} \leq q_{k_1} \leq \dots$;
- **wqo(anti)** if it is well-founded and every antichain is finite;
- **wqo(ext)** if every linear extension of Q is well-founded;
- **wqo(fbp)** if every closed subset is finitely generated;
- **wqo(acc)** if the set of closed subsets is Noetherian;
- **wqo(*)** if the set of finitely generated closed subsets is *Noetherian.

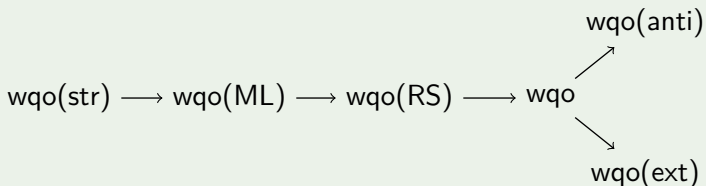
Remark: all the wqo definitions are **classically** equivalent.

Constructive relations between wqo definitions

Theorem

The conditions $wqo(\text{set})$, $wqo(\text{fbp})$ and $wqo(\text{acc})$ are not constructively meaningful.

Implications between constructive wqo definitions



A closure property

Let \mathcal{P} any of the properties wqo , $wqo(\text{anti})$, \dots except $wqo(\text{ext})$.
If (Q, \leq) has property \mathcal{P} and $P \subseteq Q$, then (P, \leq) has property \mathcal{P} .

Future work

Well-founded vs. hereditarily well-founded

Classically equivalent, but **not** constructively.

Reverse implications

Which of the following implications can be reversed?

- strongly Noetherian \Rightarrow ML-Noetherian;
- $wqo(RS) \Rightarrow wqo$;
- $wqo \Rightarrow wqo(\text{anti})$;
- ...

For now, RS-Noetherian $\not\Rightarrow$ ML-Noetherian by A. Blass.

Further closure properties

Is $wqo(\text{ext})$ closed under subset?

If P and Q have property \mathcal{P} , does

- $P \dot{\cup} Q$ constructively have property \mathcal{P} ?
- $P \times Q$ constructively have property \mathcal{P} ?

Thank you!

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