Extensions of the point to set principle to finite-state dimension

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Where do I come from, where am I going to?

1. Effectivize a notion so that it is useful in a computably-defined world
2. Use the effective notion to prove results in the classical world
3. Effectivize it some more so that you can use it in a finite-automata defined world $\rightarrow$ Use the finite-state notion to prove results in the classical world

Effective Hausdorff dimension: Kolmogorov complexity

- Given U a universal Turing Machine, and $\sigma \in 2^{<\omega}$, $K_U(\sigma)$ is the Kolmogorov complexity of σ, which is **the length of the shortest description of** σ (from which U recovers σ):

$$\mathrm{K}_U(\sigma) = \min\{|p| \ |U(p) = \sigma\}$$

- This concept is invariant on U up to an additive constant, we drop the U

$$\mathrm{K}(\sigma) = \min\{|p| \ |U(p) = \sigma\}$$

Effective Hausdorff dimension: Cantor space

Theorem (M 2002)

For $x \in 2^\omega$,

$$\dim(x) = \liminf_n \frac{\mathrm{K}(x[1..n])}{n}$$

It extends the notion of Martin-Löf random sequence:
x is ML-random iff there is a c such that for all n,

$$\mathrm{K}(x[1..n]) > n - c$$

For a set $E \subseteq 2^\omega$,

$$\dim(E) = \sup_{x \in E} \dim(x)$$

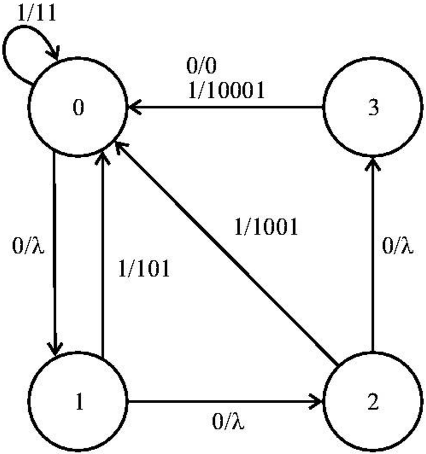
Why do we effectivize?

- To quantify
- Partial randomness
- Geometric measure theory (correspondence principles)

Sample results

- (Hitchcock 2005) If E is a union of Π^0_1-definable sets then $\dim_{\mathrm{H}}(E) = \dim(E)$
- There are Δ^0_2-degrees of dimension 1 with no ML-random reals

Most extreme effectivitation: Finite-state dimension



Given a finite-state transducer D with input and output binary alphabet (2-FST),

$$\mathrm{K}_D(\sigma) = \min \{ |p| \mid D(p) = \sigma \lor p = \sigma \}$$

Most extreme effectivitation: Finite-state dimension

Theorem (Doty Moser 2006)

For $x \in 2^\omega$,

$$\dim^2_{\mathrm{FS}}(x) = \inf_{D2\mathrm{-FST}} \liminf_n \frac{\mathrm{K}_D(x[1..n])}{n}$$

For a different input alphabet $x \in \{0, \ldots, b-1\}^\omega$

$$\dim^b_{\mathrm{FS}}(x) = \inf_{D\mathrm{b-FST}} \liminf_n \frac{\mathrm{K}_D(x[1..n])}{n}$$

For a set $E \subseteq \{0, \ldots, b-1\}^\omega$,

$$\dim^b_{\mathrm{FS}}(E) = \inf_{D\mathrm{b-FST}} \sup_{x \in E} \liminf_n \frac{\mathrm{K}_D(x[1..n])}{n}$$

Sample result

Theorem

(Lutz M 2021) There is an algorithm that computes an absolutely normal real number in nearly-linear time

Effective Hausdorff dimension in Euclidean space

We identify $x \in 2^\omega$ with the real number with binary representation $0.x$ (also denoted x)
For $x \in [0,1]$,

$$\dim(x) = \liminf_n \frac{\mathrm{K}(x[1..n])}{n}$$

At Finite-State level, **the alphabet matters**, so for $b \in \mathbb{N}$ we identify $x \in \{0, \ldots, b-1\}^\omega$ with the real number in base b, $0.x$

$$\dim_{\mathrm{FS}}^b(x) = \inf_{D \mathrm{b-FST}} \liminf_n \frac{\mathrm{K}_D(x[1..n])}{n}$$

Effective dimension in Euclidean space: adding geometry

Definition (Kolmogorov complexity of x at precision δ)

$$\mathrm{K}_\delta(x) = \inf\{\mathrm{K}(\sigma) \mid |x - 0.\sigma| < \delta\}$$

For $x \in [0,1]$,

$$\dim(x) = \liminf_{\delta \to 0^+} \frac{\mathrm{K}_\delta(x)}{\log(1/\delta)}$$

Using information content at precision δ for FS

For D a finite-state transducer with input and output alphabet $\{0, \ldots, b-1\}$ ($b \in \mathbb{N}$), for $x \in [0,1]$,

$$\mathrm{K}_{D,\delta}(x) = \inf \{\mathrm{K}_D(\sigma) \mid |x - 0.\sigma| < \delta\}$$

Theorem (M 2022)

For $x \in [0,1]$,

$$\dim^b_{\mathrm{FS}}(x) = \inf_{D\mathrm{b-FST}} \liminf_{\delta \to 0^+} \frac{\mathrm{K}_{D,\delta}(x)}{\log(1/\delta)}$$

Effective Hausdorff dimension in other separable metric spaces

Let (X, ρ) be a separable metric space and let $D \subseteq X$ be a countable dense set (fix $f : 2^{<\omega} \twoheadrightarrow D$)

Definition (Kolmogorov complexity of x at precision δ)

$$\mathrm{K}_\delta(x) = \inf \{\mathrm{K}(\sigma) \mid \rho(x, f(\sigma)) < \delta\}$$

Definition (Lutz et al 2022)

The *algorithmic dimension* of a point $x \in X$ is

$$\dim(x) = \liminf_{\delta \to 0^+} \frac{\mathrm{K}_\delta(x)}{\log(1/\delta)}$$

Using information content at precision δ for FS

For D a finite-state transducer with input and output alphabet $\{0, \ldots, b-1\}$ ($b \in \mathbb{N}$), $x \in X$,

$$\mathrm{K}_{D,\delta}(x) = \inf \{\mathrm{K}_D(\sigma) \,|\, \rho(x, f(\sigma)) < \delta\}$$

$$\dim_{\mathrm{FS}}^{b}(x) = \inf_{D\mathrm{b-FST}} \liminf_{\delta \to 0^+} \frac{\mathrm{K}_{D,\delta}(x)}{\log(1/\delta)}$$

What next?

The relativization ingredient

$$\mathrm{K}^A(\sigma) = \min\left\{|p| \,\middle|\, U^A(p) = \sigma\right\}$$

$$\mathrm{K}^A_\delta(x) = \inf\left\{\mathrm{K}^A(\sigma) \mid \rho(x, f(\sigma)) < \delta\right\}$$

$$\dim^A(x) = \liminf_{\delta \to 0^+} \frac{\mathrm{K}^A_\delta(x)}{\log(1/\delta)}$$

And for a set $E \subseteq X$

$$\dim^A(E) = \sup_{x \in E} \dim^A(x)$$

Hausdorff definition of dimension (1919)

Let (X, ρ) be a separable metric space

- For $E \subseteq X$ and $\delta > 0$, a δ-cover of E is a countable collection $\mathcal{U}$ such that for all $U \in \mathcal{U}$, $\mathrm{diam}(U) < \delta$ and
$$E \subseteq \bigcup_{U \in \mathcal{U}} U$$
- For $s \geq 0$,
$$H^s(E) = \lim_{\delta \to 0} \inf_{\mathcal{U} \text{ is a } \delta\text{-cover of } E} \sum_{U \in \mathcal{U}} \mathrm{diam}(U)^s$$

The Hausdorff dimension of $E \subseteq X$ is
$$\dim_{\mathrm{H}}(E) = \inf \{ s \,|\, H^s(E) = 0 \}$$

Point-to-set principle

Theorem (Lutz Lutz 2018, Lutz et al 2022)

Let $E \subseteq X$, then

$$\dim_{\mathrm{H}}(E) = \min_{A \subseteq 2^{<\omega}} \dim^{A}(E)$$

Two possible directions:

- Use the point-to-set principle to prove results in geometric measure theory
- Analyze the point-to-set principle to understand effectivizations of dimension

Application of point to set principles to fractal geometry: projection formula

Theorem (Marstrand 1954, Mattila 1975)

Let $E \subseteq \mathbb{R}^n$ be an **analytic set** *with* $\dim_{\mathrm{H}}(E) = s$. *Then for almost every* $e \in S^{n-1}$, $\dim_{\mathrm{H}}(p_e E) = \min\{s, 1\}$

It does not hold for arbitrary E (assuming CH). Recently an extension using PSPs

Theorem (N.Lutz Stull 2018)

Let $E \subseteq \mathbb{R}^n$ be an **arbitrary set with** $\dim_{\mathrm{H}}(E) = \dim_{\mathrm{P}}(E) = s$. *Then for almost every* $e \in S^{n-1}$, $\dim_{\mathrm{H}}(p_e E) = \min\{s, 1\}$

Hausdorff optimal oracles (Stull 2022)

(Informal) A is an Hausdorff optimal oracle for E if $\dim_{\mathrm{H}}(E) = \dim^{A}(E)$ and any oracle A, B does not decrease $\dim^{A,B}(x)$ for some $x \in E$

Theorem (Stull 2022)

Let $E \subseteq \mathbb{R}^n$ be a set that has a Hausdorff optimal oracle. Then for almost every $e \in S^{n-1}$, $\dim_{\mathrm{H}}(p_e E) = \min\{\dim_{\mathrm{H}}(E), 1\}$

All known cases of the projection theorem are particular cases of this

Revisiting the PTSPs

Let $D \subseteq X$ be a countable dense set, let us consider **different enumerators** $f : 2^{<\omega} \twoheadrightarrow D$

$$\mathrm{K}^f_\delta(x) = \inf \{ \mathrm{K}(\sigma) \,|\, \rho(x, f(\sigma)) < \delta \}$$

Definition

The *algorithmic dimension* of a point $x \in X$ with enumerator f is

$$\dim^f(x) = \liminf_{\delta \to 0^+} \frac{\mathrm{K}^f_\delta(x)}{\log(1/\delta)}$$

Revisiting the PTSPs

Theorem (M 2022)

Let $E \subseteq X$. Then

$$\dim_{\mathrm{H}}(E) = \min_{f:2^{<\omega} \twoheadrightarrow D} \dim^{f}(E).$$

Some consequences

- Relativization can be substituted by dense set enumeration
- This is a robust alternative to relativization for Finite-State dimension
- for each enumeration f we can have a robust definition of finite-state dimension $\dim^{f}_{\mathrm{FS}}$

$$\dim^{f}_{\mathrm{FS}}(x) = \inf_{D2-\mathrm{FST}} \liminf_{\delta \to 0^+} \frac{\mathrm{K}^{f}_{D,\delta}(x)}{\log(1/\delta)}$$

Finite-State dimension PTSP

Theorem (M 2022)

Let $E \subseteq [0,1]$*.*

$$\dim_{\mathrm{H}}(E) = \min_{f:2^{<\omega} \twoheadrightarrow D} \dim^{f}_{\mathrm{FS}}(E).$$

What we can learn from this

- The oracle for which $\dim_{\mathrm{H}}(E) = \min_{A \subseteq 2^{<\omega}} \dim^A(E)$ requires a single (functional) query
- It can be interesting to separate compression and relativization
- The concept of optimal oracles from (Stull 2022) should be revisited for optimal enumerators

Further directions

- For computability: Classification of PSP enumerators/oracles of a set
- For geometric m.t.: Can sets with optimal enumerators/oracles replace analytic sets in different known results?

References on the point to set principle

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