Direct Construction of Scott Ideals

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Scott ideals

Definition

In the upper semi-lattice $\mathfrak{D}$ of Turing degrees under Turing reducibility $\leq$, a *Turing ideal* T is a set closed under join and downwards under $\leq$.

Examples: the *principal* Turing ideals $T_{\boldsymbol{b}} = \{\boldsymbol{d} \in \mathfrak{D} : \boldsymbol{d} \leq \boldsymbol{b}\}$.

A degree $\boldsymbol{d}$ is *PA relative to* $\boldsymbol{c}$ if every infinite $\boldsymbol{c}$-computable subtree of $2^{<\omega}$ has a $\boldsymbol{d}$-computable (infinite) path. (The PA-degrees relative to $\mathbf{0}$ are precisely the degrees of complete extensions of Peano Arithmetic.)

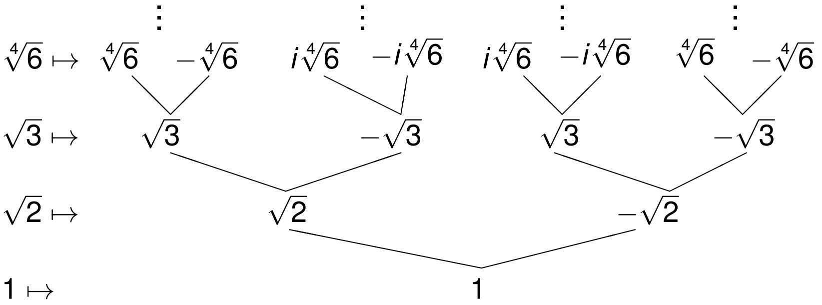
Definition

A Turing ideal T is a *Scott ideal* if, for every $\boldsymbol{c} \in T$, T also contains a degree $\boldsymbol{d}$ that is PA relative to $\boldsymbol{c}$.

No principal Turing ideal is a Scott ideal, but by closing under the PA-requirement, one can readily extend a principal Turing ideal to a countable Scott ideal – if you have a reason to want to do so.

Why Scott ideals?

My current project involves computability on $\mathrm{Aut}(\overline{\mathbb{Q}})$, the absolute Galois group of $\mathbb{Q}$. We view automorphisms of (a computable copy of) the algebraic closure $\overline{\mathbb{Q}}$ as paths through a computable tree $T_{\overline{\mathbb{Q}}}$:



Given paths f and g, one can uniformly compute $g \circ f$ and f^{-1} using Turing functionals. So this is a very nice tree presentation of $\mathrm{Aut}(\overline{\mathbb{Q}})$.

Computable automorphisms of $\overline{\mathbb{Q}}$

The subgroup $\text{Aut}_{\mathbf{0}}(\overline{\mathbb{Q}})$ of all computable automorphisms is dense in $\text{Aut}(\overline{\mathbb{Q}})$, and seems like a fairly good "effectively profinite" version of $\text{Aut}(\overline{\mathbb{Q}})$. But is it an elementary subgroup of $\text{Aut}(\overline{\mathbb{Q}})$?

Specifically for Σ_1 formulas: if $f \in \text{Aut}_{\mathbf{0}}(\overline{\mathbb{Q}})$ and

$$\text{Aut}(\overline{\mathbb{Q}}) \models (\exists G)\ G \circ G = f,$$

must $\text{Aut}_{\mathbf{0}}(\overline{\mathbb{Q}})$ contain some g with $g \circ g = f$?

To try to find such a g, one considers the subtree T of all nodes $\gamma \in \text{Aut}(F)$ such that $\gamma \circ \gamma = f \restriction F$, for finite Galois extensions $F/\mathbb{Q}$. The paths through this subtree T are precisely the solutions g. (This is the topic of ongoing work. Kundu and I have shown that there is no Φ such that, whenever $f \in \text{Aut}_{\mathbf{0}}(\overline{\mathbb{Q}})$ is a square, Φ^f is such a g.)

Subgroups defined by Scott ideals

It remains open to what extent $\mathrm{Aut}_{\mathbf{0}}(\overline{\mathbb{Q}})$ is elementary within $\mathrm{Aut}(\overline{\mathbb{Q}})$. However, using Scott ideals, we can prove some positive results.

Definition

For each Scott ideal I in the Turing degrees, set

$$\mathrm{Aut}_I(\overline{\mathbb{Q}}) = \{f \in \mathrm{Aut}(\overline{\mathbb{Q}}) : \deg(f) \in I\}.$$

Theorem (M., submitted)

Every $\mathrm{Aut}_I(\overline{\mathbb{Q}})$ as defined above is elementary within $\mathrm{Aut}(\overline{\mathbb{Q}})$ for all positive sentences, all existential and universal sentences, and for a larger class known as the Σ_2-*separated sentences*.

This theorem could yet extend to more complex sentences; again, this is current work. But it appears that Scott-ideal subgroups $\mathrm{Aut}_I(\overline{\mathbb{Q}})$ are closer to elementary in $\mathrm{Aut}(\overline{\mathbb{Q}})$ than ordinary principal-ideal subgroups $\mathrm{Aut}_{\boldsymbol{d}}(\overline{\mathbb{Q}})$ are.

Superapproximations

Since $\mathrm{Aut}_I(\overline{\mathbb{Q}})$ is apparently a nicer subgroup of $\mathrm{Aut}(\overline{\mathbb{Q}})$ than $\mathrm{Aut}_{\boldsymbol{a}}(\overline{\mathbb{Q}})$ is, we consider how, given an arbitrary set A of arbitrary degree $\boldsymbol{a}$, to construct a (countable) Scott ideal containing $\boldsymbol{a}$. The scheme is to find a set A_1 of degree $\boldsymbol{a}_1$ that is PA relative to $\boldsymbol{a}_0 = \boldsymbol{a}$, then continue with $\boldsymbol{a}_2, \boldsymbol{a}_3, \ldots$ and set $I_A = \bigcup_n T_{\boldsymbol{a}_n}$. But clearly each $\boldsymbol{a}_{n+1} > \boldsymbol{a}_n$, so we cannot just compute these from A.

Definition

A *superapproximation* h of a set B is a total function $h : \omega^2 \to 2$ such that, for all e,

$$\lim_s h(e, s) = \chi_{B'}(e).$$

B is *A-superapproximable* if A computes such an h.

If A superapproximates B, then B' is the limit of the A-computable h, and so $B' \leq_T A'$. Thus an A-superapproximation is a construction of some B that is low relative to A, though not necessarily $\leq_T A$.

Uniform Low Basis Thm. (Brattka-de Brecht-Pauly)

Theorem (B-deB-P, strengthening Jockusch & Soare)

There is a Turing functional Ψ such that, for every oracle set $S \subseteq \omega$ and every e such that Φ_e^S decides an infinite subtree T of $T_{\overline{\mathbb{Q}}}$, $\Psi^S(e, x, s)$ is a total function of x and s and there exists a path P through T for which

$$\lim_{s \to \infty} \Psi^S(e, x, s) = \chi_{P'}(x).$$

Thus Ψ^S superapproximates this path P, uniformly in S.

T may be chosen, uniformly in S, to be a tree all of whose paths have PA-degree relative to S. Thus we have a uniform way of superapproximating a set of PA-degree relative to (an arbitrary) S.

Building the Scott ideal

Now, given $A_0 = A$, we may uniformly construct an ascending sequence

$$A_0 <_T A_1 <_T A_2 <_T \cdots$$

with each A_{n+1} of PA-degree over A_n, but with

$$\cdots \leq_T A_2' \leq_T A_1' \leq_T A_0' = A'.$$

Indeed these latter reductions may be given uniformly, by the Uniform Low Basis Theorem. So the Turing ideal

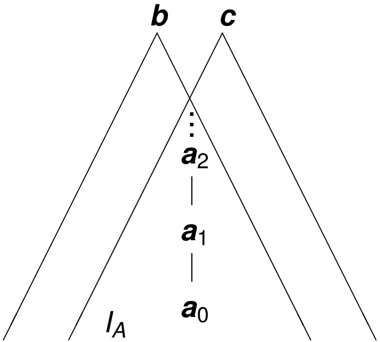
$$I_A = \bigcup_n T_{\boldsymbol{a}_n}$$

is a Scott ideal containing $\boldsymbol{a} = \deg(A)$, in which every degree $\boldsymbol{a}_n = \deg(A_n)$ is low relative to $\boldsymbol{a}$ in a uniform way.

Describing the Scott ideal I_A

Spector showed that every ascending sequence $\boldsymbol{a}_0 < \boldsymbol{a}_1 < \cdots$ of Turing degrees has an *exact pair*: a pair of degrees $\boldsymbol{b}$ and $\boldsymbol{c}$ for which

$$\{\boldsymbol{d} : (\exists n)\ \boldsymbol{d} \leq \boldsymbol{a}_n\} = \{\boldsymbol{d} : \boldsymbol{d} \leq \boldsymbol{b}\ \&\ \boldsymbol{b} \leq \boldsymbol{c}\}.$$



So $I_A = T_{\boldsymbol{b}} \cap T_{\boldsymbol{c}}$ is a *semiprincipal* Turing ideal.

A uniform exact pair

Theorem (M., following Spector; see LNCS proceedings)

There is a uniform procedure that, for every set $A \subseteq \omega$, computes superapproximations of two sets B_A and C_A such that $\{\boldsymbol{d} : \boldsymbol{d} \leq \boldsymbol{b} \ \& \ \boldsymbol{d} \leq \boldsymbol{c}\}$ is a Scott ideal containing $\deg(A)$.

The procedure uses the Uniform Low Basis Theorem as above to compute approximations to A'_0, then to A'_1, then A'_2, and so on for all A'_n, uniformly in A and n. From these A-computable approximations, it then applies Spector's method (as described and slightly updated in Soare's text) to compute approximations of the sets B_A and C_A described by Spector. Spector had already noted that B_A and C_A lie below $(\oplus_n A_n)'$; we add the superapproximability of their join here, making $(B_A \oplus C_A)' \leq_T A'$ uniformly in A.

A direct limit

$\mathrm{Aut}(\overline{\mathbb{Q}})$ is traditionally regarded as a profinite group, the inverse limit of finite Galois groups over increasing number fields, and this is the approach used to build its presentation here. But with computability, it may also be viewed as a direct limit

$$\mathrm{Aut}(\overline{\mathbb{Q}}) = \bigcup_{\boldsymbol{d} \in \mathfrak{D}} \mathrm{Aut}_{\boldsymbol{d}}(\overline{\mathbb{Q}})$$

of countable subgroups. The directed system on these groups is simply inclusion, which corresponds to Turing reducibility:

$$\mathrm{Aut}_{\boldsymbol{c}}(\overline{\mathbb{Q}}) \subseteq \mathrm{Aut}_{\boldsymbol{d}}(\overline{\mathbb{Q}}) \iff \boldsymbol{c} \leq \boldsymbol{d}.$$

Another direct limit

Since $\mathrm{Aut}_{I_{\boldsymbol{a}}}(\overline{\mathbb{Q}})$ is likely a more elementary subgroup of $\mathrm{Aut}(\overline{\mathbb{Q}})$ than $\mathrm{Aut}_{\boldsymbol{a}}(\overline{\mathbb{Q}})$ is, one naturally also considers the direct limit of these subgroups:

$$\mathrm{Aut}(\overline{\mathbb{Q}}) = \bigcup_{A \subseteq \omega} \mathrm{Aut}_{I_A}(\overline{\mathbb{Q}}).$$

The fact that these subgroups are uniformly superapproximated makes this idea seem more feasible. But there's a catch: what is the directed system here? Of course we want to use inclusion maps. But when does $\mathrm{Aut}_{I_A}(\overline{\mathbb{Q}}) \subseteq \mathrm{Aut}_{I_B}(\overline{\mathbb{Q}})$?

Question

Assume that $A \leq_T B$. Does it follow for our uniformly superapproximated Scott ideals that $I_A \subseteq I_B$?

Monotonicity and the Uniform Low Basis Theorem

Definition

A function $F : 2^\omega \to 2^\omega$ is *monotonic* if, for all A and B,

$$A \leq_T B \implies F(A) \leq_T F(B) \qquad [\iff ((F(A))' \leq_1 (F(B))'].$$

We would have a positive answer above if the ULBT is monotonic.

Question

Does there exist a monotonic, uniformly superapproximable function $F : 2^\omega \to 2^\omega$ such that, for every A, $F(A)$ has PA-degree relative to A?
In particular, does the superapproximation given by Brattka-de Brecht-Pauly satisfy monotonicity?

If such a function exists, we would apply it to build the sequence $A_0 <_T A_1 <_T \cdots$ as before, knowing that

$$A \leq_T B \implies (\forall n) A_n \leq_T B_n \implies I_A \subseteq I_B.$$