On a First-Order Theory of Building Blocks and its Relation to Arithmetic and Set Theory

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Motivation

Question

If we are given some primitive objects, just let us regard these objects as building blocks, which essential properties must they satisfy in order to encode basic finitary mathematics?

Atoms

There is an unlimited supply of atoms, in the sense that for any one thing we can always find an atom that is not a part of that thing.

Composition

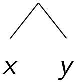
Any two things x and y can be combined to form a thing z whose parts are exactly z, the parts of x and the parts of y.

First-Order Theory BB

- Language is $\{\preceq\}$.
- Let BB^- be defined by the non-logical axioms

$$\mathrm{BB}_1 \quad \forall x \, \exists y \, [\, y \not\preceq x \;\wedge\; \forall z \, [\, z \preceq y \leftrightarrow z = y \,] \,]$$
$$\mathrm{BB}_2 \quad \forall xy \, \exists z \, \forall w \, [\, w \preceq z \leftrightarrow (\, w = z \;\vee\; w \preceq x \;\vee\; w \preceq y \,) \,]$$

- BB is BB^- extended with three axioms stating that $\preceq$ is a partial order (reflexive, symmetric, transitive).
- BB and BB^- are mutually interpretable.
- When reasoning in an arbitrary model, if $x \not\preceq y$ and $y \not\preceq x$, we visualize the composition of x and y as the unordered binary tree



- If $x \preceq y$, then we can take y as the composition of x and y.

Ordered Pairs

- We show how to define ordered pairs in BB.
- We want a relation $(x, y) \simeq z$ such that

$$\forall xy\ \exists z\ [\ (x, y) \simeq z\ ]$$

and

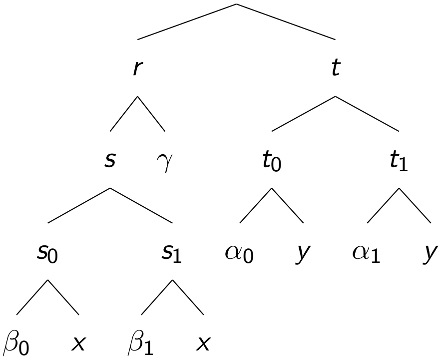
$$(x, y) \simeq z\ \wedge\ (u, v) \simeq z \rightarrow x = u\ \wedge\ y = v$$

- BB^- does not have pairing (for all elements of the universe).

- Let x and y be given.
- Choose five distinct atoms

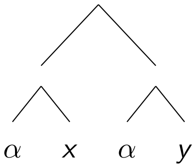
$$\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma$$

that are parts of neither x nor y, and then

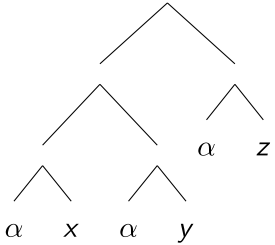


Encoding Finite Sets

- Encoding the finite set $\{x, y\}$: we choose an atom α that is a part of neither x nor y, and then



- Encoding the finite set $\{x, y, z\}$: we choose an atom α that is a part of neither x nor y nor z, and then



Interpretability Results

Theorem

BB^{-}, BB, *Adjunctive Set Theory and Robinson arithmetic are mutually interpretable.*

- An interpretation $K : S \to T$ is a uniform internal model construction $K_* : \mathrm{Mod}(T) \to \mathrm{Mod}(S)$. The universe of $K_*(\mathcal{M})$ is $X/\sim$ where $X \subseteq M$.
- K is direct if the universe of $K_*(\mathcal{M})$ is M (no relativization and equality is not redefined).

Theorem

BB *and Adjunctive Set Theory are mutually directly interpretable.*

Open Questions

Problem

Are BB *and Adjunctive Set Theory synonymous or bi-interpretable?*

Problem

What is a natural extension of Adjunctive Set Theory that is bi-interpretable with BB*?*

Thank You!