An $O(\sqrt{k})$ -approximation algorithm for minimum power k edge-disjoint st-paths

Zeev Nutov, The Open University of Israel

Network Design Problems

Input: A graph $G = (V, E)$ with edge/node costs.

Output: A min-cost subgraph H of G that satisfies a given property.

Examples of properties

■ H has minimum degree 1	Edge-Cover
■ H is connected	Minimum Spanning Tree
■ H contains an st-path	Shortest Path
■ H contains k disjoint st-paths	Min-Cost k-Flow
■ Etc.	many other problems

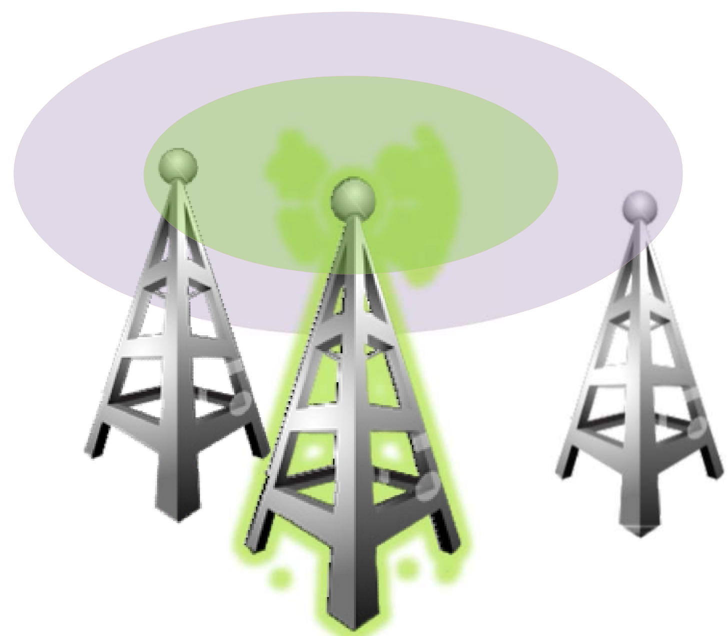
Wireless Networks

The uses of wireless networks have grown significantly in the past several decades.

Wired versus Wireless

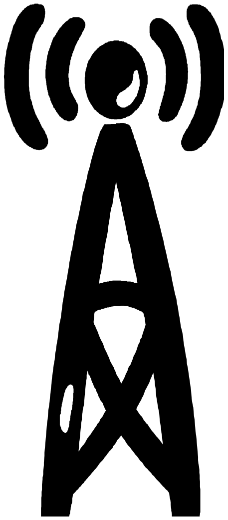
Wired Networks
connecting two nodes incurs a certain cost.

Wireless Networks
We pay at a node (transmitter) for a **range**, to connect to all nodes in the range.



Minimum Power Problems

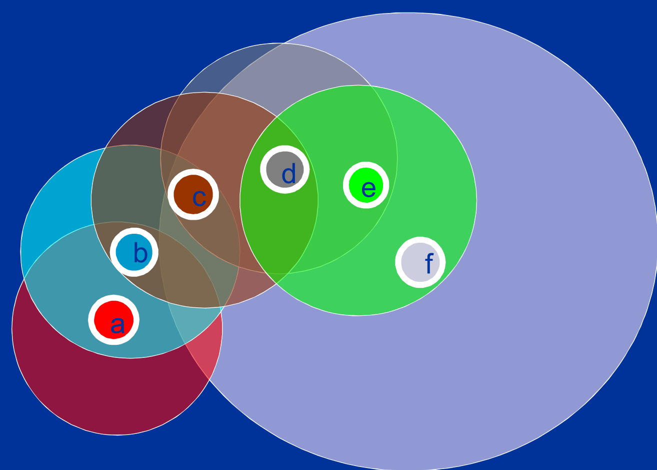
- Nodes in the network are transmitters.
- Every node connects to all nodes in its range.
- More power $\Rightarrow$ larger transmission range.
- Transmission range: a disk centered at the node.



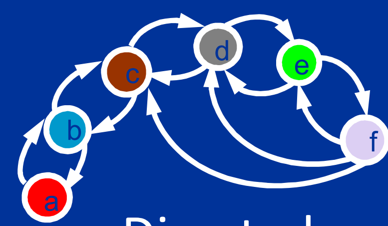
Goal: Assign **energy levels** $\{w(v): v \in V\}$ to the nodes such that:

- the **communication network** satisfies a given property;
- the **total energy** $\sum_{v \in V} w(v)$ is minimal.

Example



Range assignment



Directed

Communication network

Undirected

The Min-Power k Edge Disjoint st-Paths Problem

Min-Power k-EDP

Input: A graph $G = (V, E)$ with edge-costs $c(e)$, $s, t \in V$, integer k.

Output: An edge set $F \subseteq E$ that contains k edge-disjoint st-paths.

Minimize: $p(F) = \sum_{v \in V} w_F(v)$, where $w_F(v) = \max_{uv \in F} c(uv)$.

total power of F — **power of F at v**

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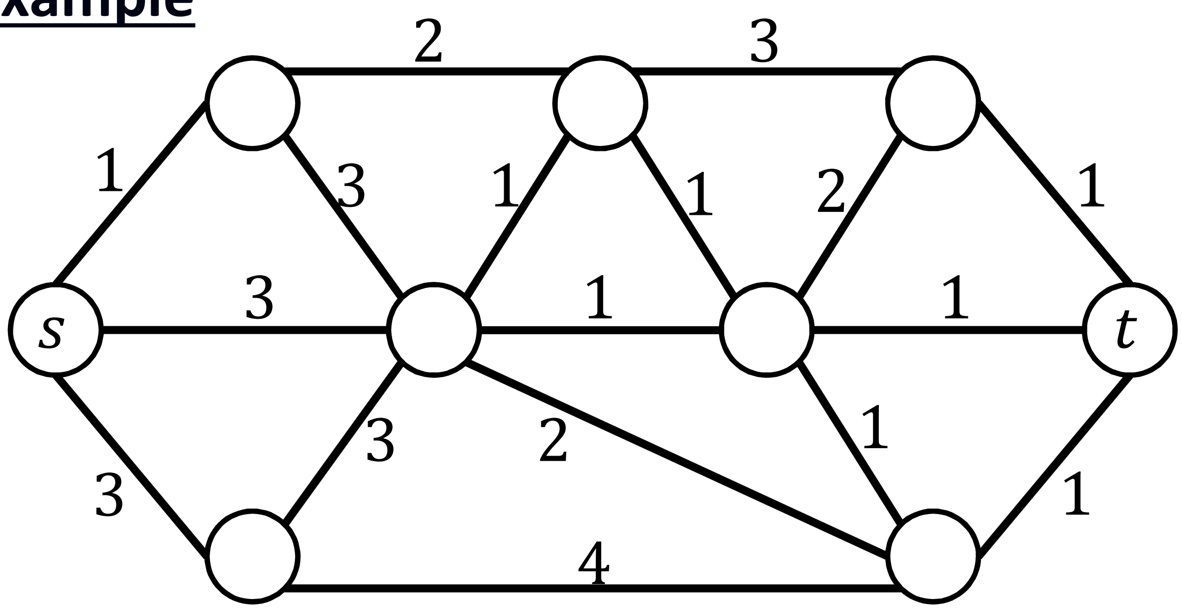

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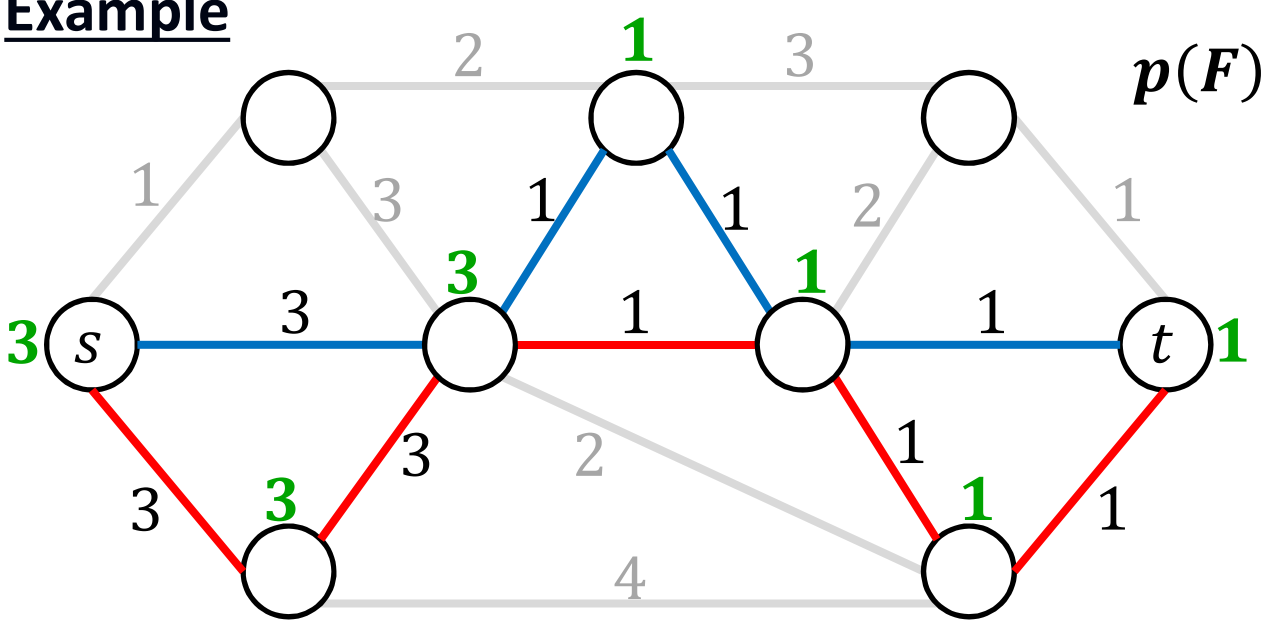

$$p(F) = 3 \cdot 3 + 1 \cdot 4$$

Relation to the Node-Weighted k-EDP problem

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Node-Weighted k-EDP

Node weights $w(v)$ instead of edge costs.

Minimize the weight $w\big(V(F)\big) = \sum_{v \in V(F)} w(v)$

of the set $V(F)$ of end-nodes of the edges in F.

Observation: Node-Weighted k-EDP with unit node weights is equivalent to Min-Power k-EDP with unit costs.

What do we know about Min-Power k-EDP?

$k = 1$: Polynomial algorithm [ACMP 03], reduction to Shortest Path.

$k = 2$: We have a 2-approximation algorithm, but ….
we don't know if the problem is in P or is NPC.
The edge-disjoint and node-disjoint versions are equivalent.

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Large values of k:

- Polynomial algorithm for increasing the number of paths by 1, when G has $k - 1$ edge disjoint st-paths of cost zero [LN 10]. This implies k-approximation for Min-Power k-EDP.
- Ratio ρ for Min-Power k-EDP with unit costs implies ratio $\alpha = 2\rho^2$ for the Densest k-Subgraph problem [N 08]. Currently $\alpha = n^{1/4+\epsilon}$ [BCCFV 10], so probably $\rho = \Omega(n^{1/8})$.

Two Open Questions

Question 1: Best known ratio is k, approximation threshold is $n^{1/8}$.
Can we achieve approximation ratio sublinear in k?

Question 2: For $k = 2$:

- Is the problem in P or is it NPC?
- Can we achieve approximation ratio better than 2?

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Theorem: Min-Power k-EDP admits ratio $4\sqrt{2k} = O(\sqrt{k})$.

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Theorem: Min-Power k-EDP admits ratio $4\sqrt{2k} = O\left(\sqrt{k}\right)$,
on simple graphs.

The Algorithm: Return a set of cheapest k edge-disjoint st-paths.

How $c(F)$ and $p(F)$ are related?

Easy to see: $p(F) \leq 2c(F)$ (tight example: single edge)

Hard to see: $c(F) \leq \sqrt{2|F|} \cdot p(F)$ (tight example: clique)

If F and F^* are optimal solutions to Min-Cost and Min-Power k-EDP,

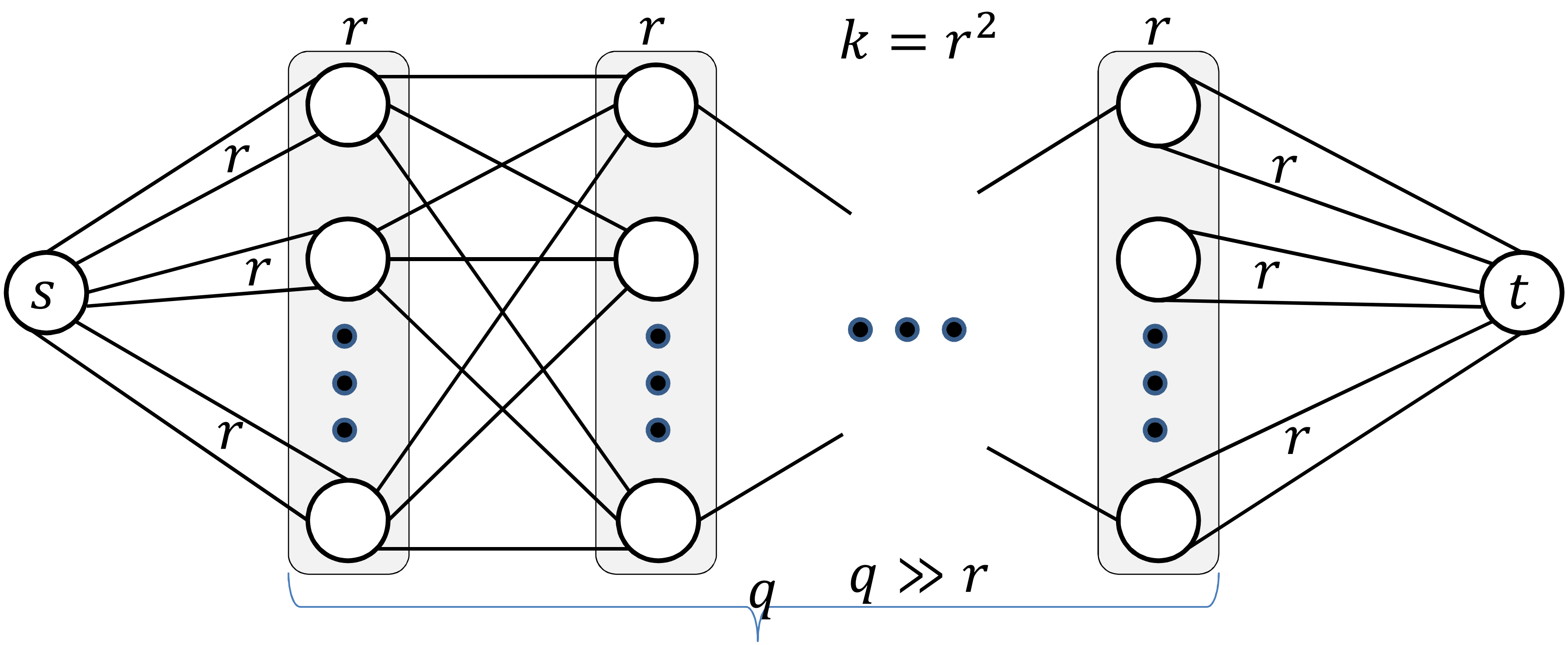
$$p(F) \leq 2c(F) \leq 2c(F^*) \leq 2\sqrt{2|F|} \cdot p(F^*)$$

Main Theorem:
If F is an inclusion minimal edge set of k edge-disjoint st-paths without parallel edges then $c(F) \leq 2\sqrt{2k} \cdot p(F)$.

Corollary: If $G = (V, E)$ is an inclusion minimal graph that contains k edge-disjoint st-paths then $|E| \leq 2\sqrt{2k} \cdot |V|$.

A tight example for unit costs

Corollary: $|E| \leq 2\sqrt{2k} \cdot |V|$.

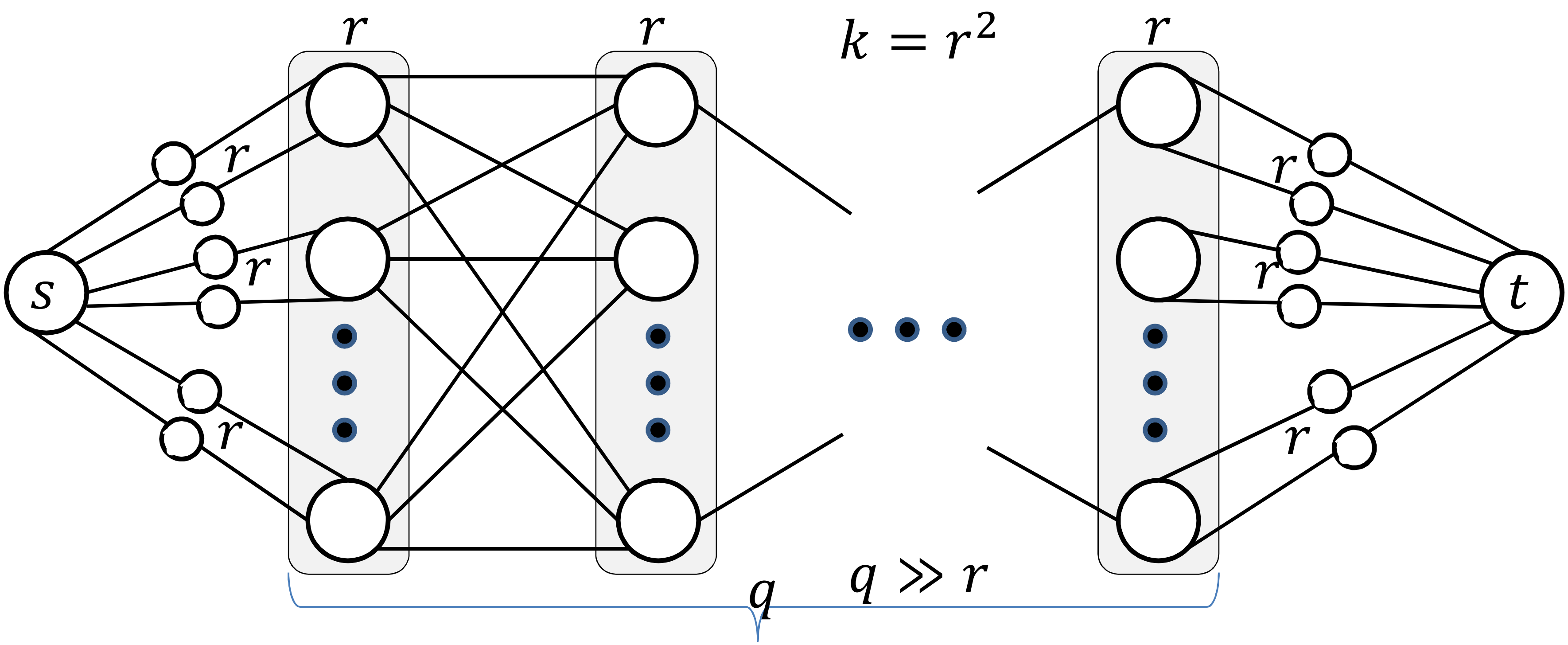


$|E| \approx r^2(q+1)$

$|V| \approx rq$

A tight example for unit costs

Corollary: $|F| \leq \mathbf{2\sqrt{2k}} \cdot |V(F)|$.



$|E| \approx r^2(q+1) + 2r(r-1)$

$|V| \approx rq + 2r(r-1)$

$$\frac{|E|}{|V|} \approx \frac{r^2(q+3)}{r(q+2r)} \approx r$$

Proof of the Theorem

Let F be an inclusion minimal edge set of k edge-disjoint st-paths without parallel edges. For $U \subseteq V$ let $F_U = \{uv \in F: u, v \in U\}$.

The main Lemma: $|F_U| \leq 2\sqrt{2k} \cdot |U|$ for any node subset U.

We use the lemma to prove that $c(F) \leq 2\sqrt{2k} \cdot p(F)$.

Let $w(v) = \max_{uv \in F} c(uv)$ be the power of F at node v. Then

$$c(F) \leq \sum_{xy \in F} \min\{w(x), w(y)\}$$

Thus it is sufficient to prove that for any node weights $w(v)$

$$\sum_{xy \in F} \min\{w(x), w(y)\} \leq 2\sqrt{2k} \cdot \sum_{v \in V} w(v)$$

Proof of the Theorem

Main Lemma: $|F_U| \le 2\sqrt{2k}\,|U|$ for any node subset U.

We need to prove: For any node weights $w(v)$

$$\sum_{xy \in F} \min\{w(x), w(y)\} \le 2\sqrt{2k} \cdot \sum_{v \in V} w(v)$$

Proof: By induction on the number N of distinct $w(v)$ values.

For $N = 1$ this follows from the Main Lemma.

For $N \ge 2$ we use “peeling”:

- $U =$ the set of max-weight nodes.
- $\epsilon =$ max-weight minus second max-weight.
- $w'(u) = w(u) - \epsilon$ for $u \in U$, $w'(u) = w(u)$ otherwise.

Proof of the Theorem

Main Lemma: $|F_U| \leq 2\sqrt{2k}\,|U|$ for any node subset U.

We need to prove: $\sum_{xy \in F} \min\{w(x), w(y)\} \leq \mathbf{2\sqrt{2k}} \cdot \sum_{v \in V} w(v)$

- U = the set of max-weight nodes.
- ϵ = max-weight minus second max-weight.
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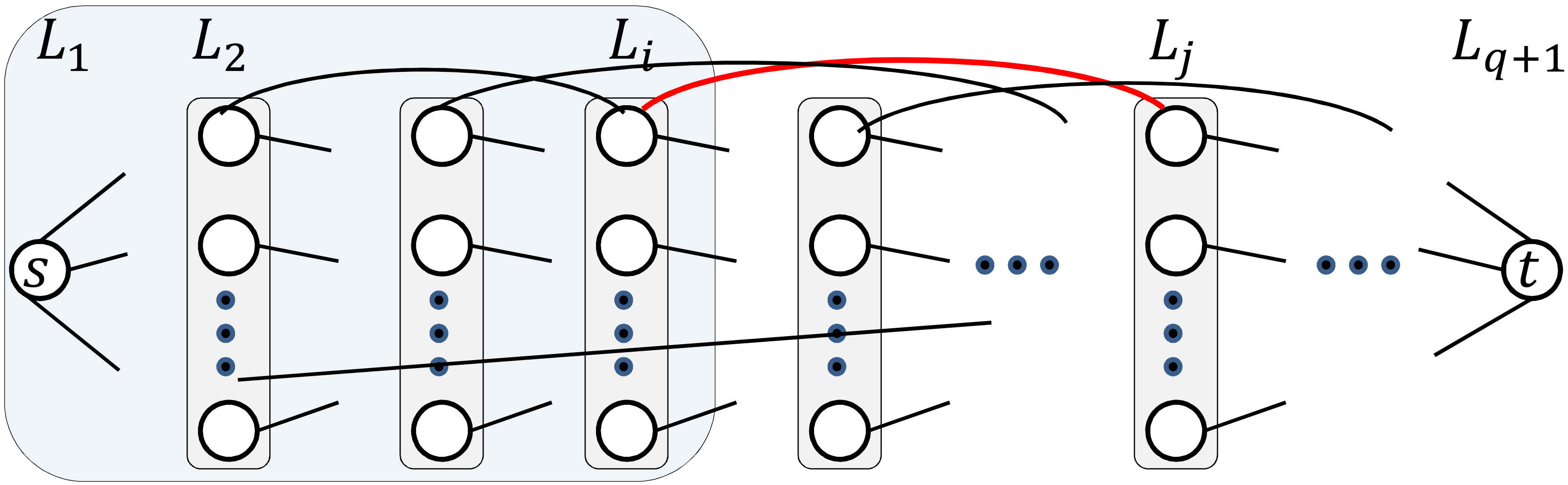
$$\sum_{xy \in F} \min\{w(x), w(y)\} = \sum_{xy \in F} \min\{w'(x), w'(y)\} + \epsilon |F_U|$$

$$\leq 2\sqrt{2k} \sum_{v \in V} w'(v) + 2\sqrt{2k} \cdot \epsilon |U|$$

$$= 2\sqrt{2k} \sum_{v \in V} w(v)$$

Proof sketch of the Main Lemma: $|E| \le 2\sqrt{2k} \cdot |V|$



1. By minimality, there exists a **nested** family of st-cuts such that: every cut has at most k edges and every edge is in some cut.
2. This partition V into layers $L_1, \dots, L_{q+1}$.
3. An edge e from L_i to L_j has length $j - i$. Let $\alpha = \sqrt{k/2}$.
4. The number of "short" edges of length $< \alpha q/n$ is at most $2\alpha n$.
5. The number of "long" edges of length $\ge \alpha q/n$ is at most kn/α.

Summary

- For Min-Power k-EDP we showed that the simplest algorithm (that computes a min-cost solution) achieves ratio $4\sqrt{2k}$.
- The proof is based on a combinatorial result – in an inclusion minimal simple graph that contains k edge-disjoint st-paths, the number of edges $\leq 2\sqrt{2k}$ the number of nodes.
- [Maier, Mecke, Wagner 07] showed that the ratio is at least $2\sqrt{k}$.

Open Questions

- Is Min-Power 2-EDP in P, or is it NPC?
- Approximation ratio better than 2 for Min-Power 2-EDP?

Thank you for attention!

Questions?