Rogers Semilattices of limitwise monotonic numberings

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Outline

- We investigate numberings for a special subclass of Σ^0_2 sets — *limitwise monotonic sets*.
- A set $A \subseteq \omega$ is *limitwise monotonic* if either $A = \emptyset$, or A is the range of a limitwise monotonic function.
- A total function $F \colon \omega \to \omega$ is *limitwise monotonic* (or *s-function*) if there is a computable function $f(x, s)$ with the following properties:
  - $f(x, s) \leq f(x, s+1)$ for all x and s, and
  - $F(x) = \lim_s f(x, s)$ for all x.

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- The notion of s-function (or limitwise monotonic) was introduced by Khisamiev[1]. He used limitwise monotonic approximations of Σ_2^0 sets to study computable abelian p-groups.
- Coles, Downey, and Khoussainov[2] used limitwise monotonic sets for building computable linear orders with Π_2^0 initial segments that are not computably presentable and first introduced the term *limitwise monotonic function*.

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- Any surjective mapping α of the set ω of natural numbers onto a nonempty set A is called a *numbering* of A.
- So numbering is the assignment of natural numbers to a set of objects such as functions, rational numbers, graphs, or words in some formal language.
- Let α and β be numberings of A. We say that a numbering α is *reducible* to a numbering β (in symbols, $\alpha \leqslant \beta$) if there exists a computable function f such that $\alpha(n) = \beta(f(n))$ for any $n \in \omega$.
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- Define some language L and the interpretation of that language determined as a partial surjective mapping $i : L \to A$. For any object $a \in A$, each "formula" in $i^{-1}(a)$ is interpreted as a description of a.
- For example, if A consists of partial computable functions then $i^{-1}(a)$ may be considered as a set of programs of Turing machines for a.
- If A is a set of c.e. sets then $a \in A$ is definable by Σ_1^0 -formulas in arithmetics and we could consider $i^{-1}(a)$ as a collection of such formulas.
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Computable Numberings

Definition

A numbering $\alpha : \omega \to A$ is called a *computable numbering* of A in the language L with respect to the interpretation i if there exists a computable function f for which the formula $G(f(n))$ distinguishes an element $\alpha(n)$ in L relative to i, i.e. $\alpha(n) = i(G(f(n)))$ for all $n \in \omega$.

- Goncharov and Sorbi[3] generalized the theory of numberings to different notions of computability:
  Let $\mathcal{C}$ be an abstract "notion" of computability, i.e., a countable class of sets of numbers, and let $\mathcal{A} \subseteq \mathcal{C}$: then a numbering $\alpha : \omega \to \mathcal{A}$ is $\mathcal{C}$-computable, if

$$\{< k, x >: x \in \alpha(k)\} \in \mathcal{C}$$

.

[3] S. Goncharov and A. Sorbi. Generalized computable numerations and non-trivial Rogers semilattices. Algebra and Logic, 36(6):359–369, 1997.

Rogers semilattices

- A family $\mathcal{A} \subset P(\omega)$ is $\mathcal{C}$-*computable* if $\mathcal{A}$ has a $\mathcal{C}$-computable numbering.
- By $Com_{\mathcal{C}}(\mathcal{A})$ we denote the set of all $\mathcal{C}$-computable numberings of the family $\mathcal{A}$.
- Given numberings ν and μ of a family $\mathcal{A}$, one defines a new numbering $\nu \oplus \mu$ as follows.

$$(\nu \oplus \mu)(2n) := \nu(n), \quad (\nu \oplus \mu)(2n+1) := \mu(n).$$

- For a computable family $\mathcal{A}$, the quotient structure

$$R_{\mathcal{C}}(\mathcal{A}) = (Com_{\mathcal{C}}(\mathcal{A}); \leq, \oplus)/\equiv$$

is an upper semilattice. It is called the Rogers semilattice [4] of the computable family $\mathcal{A}$.

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- *Rogers semilattice* $\mathcal{R}_{\mathcal{C}}(\mathcal{A})$ of a family $\mathcal{A} \subseteq \Sigma^i_n$ is a quotient structure of all $\mathcal{C}-$computable numberings of the family $\mathcal{A}$ modulo equivalence of the numberings ordered by the relation induced by reducibility of the numberings.
  - $\mathcal{R}_{\mathcal{C}}(\mathcal{A})$ allows one to measure the different computations of a given family $\mathcal{A}$.
  - It also serves as a tool to classify properties of $\mathcal{C}-$ computable numberings for the different families $\mathcal{A}$.
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Limitwise monotonic numberings

We consider *limitwise monotonic numberings* for families $\mathcal{S}$ of l.m.sets. In the definition, we use the following notation. Let $n \geq 1$, and let F be a function acting from ω^{n+1} to ω. For numbers $k_1, k_2, \ldots, k_n \in \omega$, we write $F(k_1, k_2, \ldots, k_n, \cdot)$ to denote the unary function

$$g \colon y \mapsto F(k_1, k_2, \ldots, k_n, y).$$

Definition

A numbering ν is *limitwise monotonic* if there exists a computable function $f(k, z, s)$ with the following properties:

1. $f(k, z, s) \leq f(k, z, s+1)$, for all k, z, s.
2. For every k and z, there exists a finite limit $F(k, z) = \lim_s f(k, z, s)$.
3. For every k, the set $\nu(k)$ is equal to the range of the function $F(k, \cdot)$.

Such a function f is called a *limitwise monotonic approximation* of the numbering ν.

- We say that a family $\mathcal{S}$ is *limitwise monotonic* if it has a limitwise monotonic numbering. Informally speaking, l.m. families $\mathcal{S}$ are precisely those that admit a uniform l.m. approximation.
- For a l.m. family $\mathcal{S}$, by $\mathcal{R}_{lm}(\mathcal{S})$ we denote the Rogers semilattice induced by the l.m. numberings of $\mathcal{S}$. Notice that this is consistent with the above introduced notation $\mathcal{R}_{\mathcal{C}}(\mathcal{S})$, by taking $\mathcal{C}$ to be the set of all l.m. numberings, and denoting in this case $\mathcal{C} = lm$.

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Following the approach[5], we discuss reductions between types of numberings. Our reduction Γ transforms a Σ^0_2-computable family into a l.m. family. The reduction Γ is very simple. Nevertheless, it is pretty useful: as an immediate consequence, we obtain that every Rogers Σ^0_2-semilattice $\mathcal{R}^0_2(\mathcal{S})$ is isomorphic to the semilattice $\mathcal{R}_{lm}(\Gamma(\mathcal{S}))$.

Definition

For a set $A \subseteq \omega$, we set $\Gamma(A) := A \oplus \omega$. For a numbering ν, by $\Gamma(\nu)$ we denote the following numbering: for $k \in \omega$, define

$$(\Gamma(\nu))(k) := \Gamma(\nu(k)).$$

For a family $\mathcal{S}$, we define $\Gamma(\mathcal{S}) = \{\Gamma(A) : A \in \mathcal{S}\}$.

[5] I. Herbert and S. Jain and S. Lempp and M. Mustafa and F. Stephan, Reductions between types of numberings, 2017

Theorem

Let $\mathcal{S}$ be a Σ^0_2-computable family. Then the following holds:

1. *The family $\Gamma(\mathcal{S})$ is limitwise monotonic.*
2. *The operator Γ is a bijection from the set of all Σ^0_2-computable numberings of $\mathcal{S}$ onto the set of all l.m. numberings of $\Gamma(\mathcal{S})$.*
3. *A numbering $\nu \in Com_{\Sigma^0_2}(\mathcal{S})$ is positive if and only if $\Gamma(\nu)$ is positive. A similar fact is true for Friedberg numberings.*
4. *For any $\nu, \mu \in Com_{\Sigma^0_2}(\mathcal{S})$, we have $\nu \leq \mu$ if and only if $\Gamma(\nu) \leq \Gamma(\mu)$.*

Consequently, the semilattices $\mathcal{R}^0_2(\mathcal{S})$ and $\mathcal{R}_{lm}(\Gamma(\mathcal{S}))$ are isomorphic.

In particular, this implies that there are infinitely many pairwise non-elementarily-equivalent Rogers semilattices for l.m. families.

Theorem

Suppose that $\mathcal{S}$ is a l.m. family such that every set $A \in \mathcal{S}$ is infinite. Then a numbering ν of the family $\mathcal{S}$ is limitwise monotonic if and only if ν is Σ_2^0-computable. Consequently, the semilattices $\mathcal{R}_{lm}(\mathcal{S})$ and $\mathcal{R}_2^0(\mathcal{S})$ are equal.

These two theorems allow to transfer many known results on Σ_2^0-computable families into the l.m. setting.

Corollary

There exist l.m. families $\mathcal{S}_i$, $i \in \omega$, such that the corresponding Rogers semilattices $\mathcal{R}_{lm}(\mathcal{S}_i)$ are pairwise not elementarily equivalent. Consequently, there are infinitely many isomorphism types of Rogers l.m. semilattices.

We also establish some differences between Σ^0_2-computable families and l.m. families.

- **Proposition:** For a non-zero $n \in \omega$, consider the following finite families:

$$\mathcal{S}_n = \{\emptyset, \{0\}, \{0,1\}, \{0,1,2\}, \ldots, \{0,1,2,\ldots,n-1\}\},$$
$$\mathcal{T}_n = \{\{0\}, \{1\}, \{2\}, \{3\}, \ldots, \{n\}\}.$$

The Rogers semilattices $\mathcal{R}^0_1(\mathcal{S}_n)$ and $\mathcal{R}_{lm}(\mathcal{T}_n)$ are isomorphic.

- For a non-zero number n, consider the poset P_n, which is obtained by deleting the greatest element from the finite poset $(\mathcal{S}_n, \subseteq)$.
- Notice the following:
  - if $m \neq n$, then the posets P_m and P_n are not isomorphic. By a result of Ershov [?], this implies that the semilattices $\mathcal{R}^0_1(\mathcal{S}_m)$ and $\mathcal{R}^0_1(\mathcal{S}_n)$ are not isomorphic.
  - Consequently, Proposition "transfers" infinitely many isomorphism types of Rogers Σ^0_1-semilattices into the limitwise monotonic setting.

To our best knowledge, it is still unknown whether there exists a Σ^0_2-computable family $\mathcal{T}$ such that the semilattice $\mathcal{R}^0_2(\mathcal{T})$ is isomorphic to $\mathcal{R}^0_1(\mathcal{S}_n)$ for some $n \geq 1$.

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Cardinalities and Laticeness of Rogers Semilattice of l.m.numberings

The first two problems on Rogers semilattices for computable families were raised by Ershov(1967):

(i) What are possible cardinalities of Rogers semilattices?

(ii) Can a non-one-element Rogers semilattice be a lattice?

Khutoretskii(1971) proved the following: if a Rogers semilattice $\mathcal{R}^0_1(\mathcal{S})$ contains more than one element, then it is infinite. Selivanov(1976) established that an infinite $\mathcal{R}^0_1(\mathcal{S})$ cannot be a lattice.

We consider these two problems in the limitwise monotonic setting. We obtain the following:

Theorem

Suppose that a l.m. family $\mathcal{S}$ contains at least two elements. Then the semilattice $\mathcal{R}_{lm}(\mathcal{S})$ is infinite, and it is not a lattice.

Thank you for your attention!