Cupping computably enumerable degrees simultaneously

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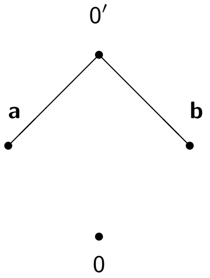
Basics:

- Computable (decidable) sets,
- Computably enumerable (semi-decidable) sets
- Turing reduction and Turing degrees
- A c.e. degree is a Turing degrees containing some c.e. sets
- C.e. degrees form a upper-semi-lattice: given c.e. degrees
  - given c.e. degrees $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \vee \mathbf{b}$, the supremum of $\mathbf{a}$ and $\mathbf{b}$ always exists (but not for $\mathbf{a} \wedge \mathbf{b}$, the infimum)
  - Friedberg-Muchinik theorem, Sacks' theorems: splitting and density, Lachlan's nonsplitting theorem
  - High/Low hierachy,
  - Cups (to $\mathbf{0}'$) and Caps (to $\mathbf{0}$), Cuppable and Cappable

Sacks' splitting and Lachlan's nonsplitting

Sacks' Splitting Theorem:

There are incomparable c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ with $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'$.



Sacks' Density Theorem:

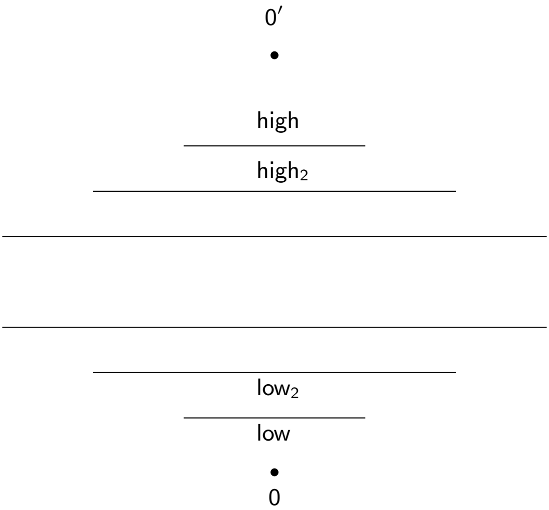
The c.e. degrees are dense.

Lachlan's nonsplitting Theorem (Harrington versioin):

There is an incomparable c.e. degrees $\mathbf{c}$ such that $\mathbf{0}'$ is not splittable above $\mathbf{c}$.

Turing Jump - $A' = \{e : \Phi_e^A(e) \text{ converges}\}$

- If $A \equiv_T B$, then $A' \equiv_T B'$.
- The jump of $\mathbf{0}$ is $\mathbf{0}'$.
- $'$ is monotonic.



Cuppable/Noncuppable

Definition:

A c.e. degree $\mathbf{a}$ is cuppable if there is an incomplete c.e. degree $\mathbf{c}$ such that $\mathbf{a} \vee \mathbf{c} = \mathbf{0}'$.

Continuity of Cupping (Ambos-Spies, Lachlan and Soare):

Given two incomplete, incomputable c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ with $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'$, there exists a c.e. degree $\mathbf{c} < \mathbf{b}$ such that $\mathbf{a} \vee \mathbf{c} = \mathbf{0}'$.

Definition:

A c.e. degree $\mathbf{a}$ is plus-cupping if every nonzero c.e. degree $\mathbf{c}$ below $\mathbf{a}$ is cuppable.

Theorem (Yates):

There are incomputable c.e. degrees which are noncuppable.

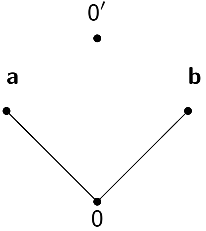
Corollary:

Plus-cupping degrees and noncuppable degrees form minimal pairs.

Here nonzero c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ form a minimal pair if $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.

Minimal pairs, Cappable/Noncappable

Lachlan and Yates first proved the existence of minimal pairs.



Continuity of Capping (Harrington and Soare):

Given two incomplete, incomputable c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ with $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$, there exists a c.e. degree $\mathbf{c} > \mathbf{b}$ such that $\mathbf{a} \wedge \mathbf{c} = \mathbf{0}$.

Definition:

A c.e. degree $\mathbf{c}$ is cappable if $\mathbf{c}$ is either $\mathbf{0}$ or a part of a minimal pair.

We have seen that plus-cupping and noncuppable degrees are cappable.

Equivalent Charaterizations: A c.e. degree $\mathbf{a}$ is noncappable if and only if one of the following is true

- $\mathbf{a}$ is low-cuppable
- $\mathbf{a}$ is promptly simple

The ideal M of Cappable degrees and the quotient structure R/M

Theorem (Ambos-Spies, Jockusch, Shore and Soare, 1984):

In R, the computably enumerable degrees, cappable degrees form an ideal and noncappable degrees form a strong filter.

Denote this ideal as M.

Consider the quotient structure R/M.

There are no minimal pairs in this structure, as noncappable degrees form a strong filter in R.

Open: dense?

Note that noncuppable degrees for another ideal in R, denoted as $NCup$.

Consider the quotient structure $R/NCup$.

Question:
Are there minimal pairs in $R/NCup$?

Answer: YES

It was first proved by Li, Wu and Yang around 2006.

But, all nonzero elements are cuppable, in $R/NCup$, of course.

Theorem (Li, Wu and Yang, 2006):

There are two incomplete cuppable degrees **a** and **b** such that no incomplete degree can cup both **a** and **b** to $\mathbf{0}'$.

That is, **a** and **b** themselves are cuppable, but they cannot be cupped to $\mathbf{0}'$ simultaneously.

- [**a**] and [**b**] form a minimal pair in $R/NCup$, as for any [**c**] below both [**a**] and [**b**], if $[\mathbf{c}] \neq [\mathbf{0}]$, then **c** is cuppable and we can let **e** incomplete with $\mathbf{e} \vee \mathbf{c} = \mathbf{0}'$, and this **e** cups both **a** and **b** to $\mathbf{0}'$, which is impossible by our choice of **a** and **b**.

Proof idea: Construct c.e. sets (incomplete incomputable) A, B, C, D and partial computable functionals Γ and Δ such that

- C cups A to $\emptyset'$ via Γ: $\emptyset' = \Gamma^{C \oplus A}$
- D cups B to $\emptyset'$ via Δ: $\emptyset' = \Delta^{D \oplus B}$

- For any c.e. W_e, if $\Phi_e^{A \oplus W_e} = \Psi_e^{B \oplus W_e} = \emptyset'$, then there is a partial computable functional Ω_e such that $\emptyset' = \Omega_e^{W_e}$.

We cannot construct A (via C) without B (via D) as we are constructing C incomplete:

- C cups A to $\emptyset'$ via Γ: $\emptyset' = \Gamma^{C\oplus A}$
- For any c.e. W_e, if $\Phi_e^{A\oplus W_e}$ $\boxed{= \Psi_e^{B\oplus W_e}}$ $= \emptyset'$, then there is a partial computable functional Ω_e such that $\emptyset' = \Omega_e^{W_e}$.

To make C incomplete, we need to enumerate numbers into A from time to time, and this can cause the computation $\Phi_e^{A\oplus W_e}(x)$ to change, and before the next agreement, x may enter $\emptyset'$, and lead to $\Omega_e^{W_e}(x) = 0 \neq \emptyset'(x)$.

Luckily, we have B-side to help us to record $\emptyset'$, to make sure that after we put a number into A, if $\emptyset'$ changes at x, W_e must have a corresponding change, as $\Psi_e^{B\oplus W_e}(x)$ changes and B has no change.

We introduce an auxiliary set L to force changes from W_e.

- If $\Phi_e^{A\oplus W_e} = \Psi_e^{B\oplus W_e} = L \oplus \emptyset'$, then there is a partial computable functional Ω_e such that $\emptyset' = \Omega_e^{W_e}$.

Theorem (with Tran):

For each $n \geq 1$, there are $n+1$ c.e. degrees $\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_n$, such that any n many of them are simultaneously cuppable, but no incomplete c.e. degree can cup all of them to $\mathbf{0}'$ simultaneously.

In this paper, we provide a detailed proof for $n = 2$:

There are c.e. degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that any two of them are simultaneously cuppable, but no incomplete c.e. degree can cup all of them to $\mathbf{0}'$ simultaneously.

More on $R/NCup$

We were able to prove that the diamond lattice can be embedded into $R/NCup$ preserving 0 and 1.

- The paper will be out soon.
- Embedding of N_5, and perhaps M_3.

Many algebraic questions are not attacked. Of course, we can consider model-theoretic properties of this structure.

Almost deep degrees form an ideal of R.

Thanks!