Multiple Permitting Notions For The Not Totally ω-C.A. Computably Enumerable Degrees

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CiE 2023 - Batumi
Special Session on Classical Theories of Degrees
24 July 2023

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Introduction

Permitting - its basic form and some extensions

What is “Permitting”?

- Permitting is a basic technique for constructing a computably enumerable (c.e.) set B which is Turing reducible to a given c.e. set A.
- This is achieved by enumerating a new number x into B at a stage $s+1$ only if (for a given enumeration of A) a number $\leq x$ (or, more generally, $\leq f(x)$ for some computable function f) enters A at stage $s+1$.

  “straight permitting” (“f-bounded permitting”)
- Obviously this ensures that $B \leq_{\mathrm{T}} A$ (in fact, $B \leq_{\mathrm{wtt}} A$, and, in case of straight permitting, even $B \leq_{\mathrm{ibT}} A$).

Typical applications of the Permitting Technique

We want to construct a c.e. set B Turing below a given c.e. set A such that B has certain properties which (in part) can be ensured by meeting positive requirements $\mathcal{P}_e$ ($e \geq 0$) of the following type:

In order to meet $\mathcal{P}_e$ it suffices to pick a follower x (becoming a witness for the fact that $\mathcal{P}_e$ will be met). The follower may become "realized" at some stage. In this case the follower (or a certain greater number) has to be enumerated into B.

In the presence of permitting, once x is realized we wait that A permits x to enter B (and, if so, we put x into B). While waiting, we iterate the attack on $\mathcal{P}_e$ with a new follower $x' > x$ (and so on).

Which sets give Permitting?

- In the above setting, any noncomputable c.e. set A will eventually permit.
- So sets with properties which can be ensured by requirements of the above described type can be found T-below (in fact, wtt-below or, in case of straight permitting, even ibT-below) any noncomputable c.e. set.

  Some classical examples:
  - For any noncomputable c.e. set A there is a simple set $B \leq_T A$.
  - For any noncomputable c.e. set A there are Turing incomparable c.e. sets $B_0, B_1 \leq_T A$ (Muchnik).
  - Any countable distributive lattice can be embedded into any proper initial segment of the c.e. degrees (by an embedding that does not preserve the least element).

Stronger forms of Permitting I: Almost-Everywhere and Prompt Permitting

- More involved positive requirements or settings require stronger forms of permitting in order to perform the construction below a given c.e. set A. These stronger permittings, however, are given not by all sets. In fact, the sets giving such stronger permittings characterize important degree classes.
  - The requirement $\mathcal{P}_e$ is *infinitary* and requires that almost all numbers in a computable ascending sequence $x_0 < x_1 < x_2 < \dots$ are enumerated into B ("almost-everywhere permitting" – Martin, Yates, Cooper)

    Corresponding permitting degrees: the high c.e. degrees (Martin)
  - The follower has to be simultaneously permitted by the set A and some other noncomputable c.e. set ("prompt permitting" – Maass).

    Corresponding permitting degrees: the noncappable c.e. degrees or, equivalently, the low-cuppable c.e. degrees (AS, Jockusch, Shore and Soare)

Stronger Forms of Permitting II: Multiple Permitting

- Any follower x of a requirement $\mathcal{P}_e$ is associated with an entourage of $\leq f(x)$ (larger) numbers (f a computable function) all of which need permitting after becoming "realized" ("multiple permitting")

  Here f may be fixed or it may depend on the requirement ($f = f_e$)

- The former case was discussed by Downey, Jockusch and Stob in 1990 (DJS1990) where it is argued that the sets and degrees giving this type of permitting are the array noncomputable (a.n.c.) sets and their degrees, respectively.

- The latter case is treated in Downey, Greenberg and Weber (2007). In contrast to DJS1990, however, only the degrees containing sets providing the desired form of multiple permitting are described but the corresponding sets are not characterized.

Here we will discuss these two types of multiple permitting.

Array noncomputability - the first formalization of multiple permitting

(The original definition, properties, and equivalent characterizations)

Array Noncomputablity - Definition

- A sequence $\mathcal{F} = \{F_n\}_{n \geq 0}$ of finite sets is a very strong array (v.s.a.) if
  (i) there is a computable function f such that $f(n)$ is the canonical index of F_n,
  (ii) $F_m \cap F_n = \emptyset$ if $m \neq n$,
  (iii) $0 < |F_n| < |F_{n+1}|$ for all $n \geq 0$, and
  (iv) $\bigcup_{n \geq 0} F_n = \omega$.

- A c.e. set A is $\mathcal{F}$-array noncomputable ($\mathcal{F}$-a.n.c.) if A is $\mathcal{F}$-similar to any c.e. set V, i.e.,

$$\exists^\infty n \, (A \cap F_n = V \cap F_n)$$

  (equivalently, it suffices to require this for a single n).

- A c.e. set A is array noncomputable (a.n.c.) if A is $\mathcal{F}$-a.n.c. for some v.s.a. $\mathcal{F}$. And A is array computable (a.c.) otherwise.

- A c.e. r-degree $\mathbf{a}$ is a.n.c. if there is an a.n.c. set in $\mathbf{a}$. And $\mathbf{a}$ is a.c. otherwise.

Why do array noncomputable sets multiply permit?

If (the entourage of) a follower x has to be permitted up to $f(x)$ times then we choose a v.s.a. $\mathcal{F}$ of intervals F_n such that $|F_n| \geq f(\min F_n)$ and choose the numbers $x = \min F_n$ as followers.

Then, whenever a member x_m of the entourage of x needs permitting, we enumerate the corresponding element y_m of F_n into a "trigger set" V. For any n sucht that A and V agree on F_n, this number y_m has to enter A later thereby giving the required (straight) permitting.

NB: Actually, the a.n.c. sets provide more than the required permittings: if we want A to change below $x + 1$ (for a straight permitting or below $f(x) + 1$ for an f-bounded permitting) after a stage s then by putting x in the trigger set V we actually achieve more, namely we make A to change *on* x. (I will come back to this.)

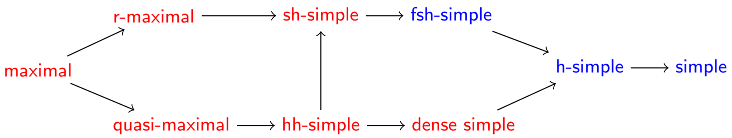
Basic Properties of the a.n.c. sets (DJS1990)

- Array invariance up to wtt-equivalence: If $\mathcal{F}$ and $\mathcal{F}'$ are very strong arrays, and A is $\mathcal{F}$-a.n.c. then there is an $\mathcal{F}'$-a.n.c. set B which is wtt-equivalent to A.
- Failure of wtt-degree invariance: For any a.n.c. set A there is a c.e. set B which is wtt-equivalent to A and not a.n.c.
- Distribution of the array noncomputable T-degrees:
  - $\mathbf{ANC}_T$ is closed upwards (in the u.s.l. of the c.e. degrees)
  - $\mathbf{NonLow}_2 \subset \mathbf{ANC}_T$
  - $\mathbf{ANC}_T$ splits the jump classes $\mathbf{Low}_2 \setminus \mathbf{Low}_1$ (Downey 1993) and $\mathbf{Low}_1$ as well as the classes **Cap** and **NonCap**
  - There are a.c. T-degrees $(\mathbf{a}_0, \mathbf{a}_1)$ s.t. $\mathbf{a}_0 \vee \mathbf{a}_1 = \mathbf{0}'$
  - No contiguous degree is a.n.c. (Downey 1993)

Simplicity Properties of the a.n.c. sets (DJS1990)

- No a.n.c. set is dense simple (hh-simple, quasi-maximal, maximal).
- No a.n.c. set is strongly h-simple (r-maximal, maximal)

  Since DJS1990 also show that there is an a.n.c. finitely strongly h-simple (hence h-simple, simple) set, the latter two results show which of the common simplicity notions are compatible with array noncomputability:



Do the above negative results on the a.n.c. sets say that the corresponding properties cannot be shared by any multiply permitting sets?

Multiply permitting vs. array noncomputability

In the abstract of DJS1990

Abstract

We study a class of permitting arguments in which each positive requirement needs multiple permissions to succeed. Three natural examples of such constructions are given. We introduce a class of r. e. sets, the array nonrecursive sets, which consists of precisely those sets which allow enough permission for these constructions be performed. We classify the degrees of array nonrecursive sets and so classify the degrees in which each of these constructions can be performed.

it is said that the a.n.c. sets are precisely those sets which give multiple permitting. As we have seen, however, the a.n.c. property might be a stronger property. (Namely when we force an a.n.c. set A to permit x, we actually force x itself into A.)

In order to decide whether the a.n.c. sets actually coincide with the multiply permitting sets, we give a more faithful formalization of the latter notion and compare it with array noncomputability.

The formal multiple permitting notion of AS2018

DEFINITION. Let A be a c.e. set, let $\{A_s\}_{s\geq 0}$ be a computable enumeration of A, let $\mathcal{F} = \{F_n\}_{n\geq 0}$ be a v.s.a., and let f be a computable function.

A is f-bounded $\mathcal{F}$-permitting via $\{A_s\}$ if, for any partial computable function ψ,

$$\exists^\infty n \,\forall\, x \in F_n \,(\psi(x)\downarrow \Rightarrow\ A \restriction f(x) + 1 \neq A_{\psi(x)} \restriction f(x) + 1).$$

So, in order to force A to give f-bounded permittings, we define a partial computable function ψ. Then, whenever we want A to f-permit some number $x \in F_n$ after some stage s_x, we let $\psi(x) = s_x$. For the infinitely many numbers n, such that A respects ψ on F_n, all corresponding requests will be fulfilled (i.e., A will change on a number $\leq f(x)$ after stage s_x).

In particular, for $f = id$, we can force A to give straight permissions. (Recall that in case of an $\mathcal{F}$-a.n.c. set we obtained the corresponding permitting by enumerating the number x in a trigger set V at stage s_x thereby forcing x into A later – provided that F_n is one of the intervals on which A and V agree.)

The formal multiple permitting notion of AS2018 (ctd.)

DEFINITION. Let A be a c.e. set, let $\{A_s\}_{s \geq 0}$ be a computable enumeration of A, let $\mathcal{F} = \{F_n\}_{n \geq 0}$ be a v.s.a., and let f be a computable function.

- A is f-bounded $\mathcal{F}$-permitting via $\{A_s\}$ if, for any partial computable function ψ,

$$\exists^\infty n \, \forall x \in F_n \, (\psi(x) \downarrow \Rightarrow \; A \upharpoonright f(x) + 1 \neq A_{\psi(x)} \upharpoonright f(x) + 1).$$

- A is f-bounded $\mathcal{F}$-permitting if A is f-bounded $\mathcal{F}$-permitting via some $\{A_s\}$; A is $\mathcal{F}$-permitting if A is f-bounded $\mathcal{F}$-permitting for some f; and A is multiply permitting if A is $\mathcal{F}$-permitting for some $\mathcal{F}$.

- If A is f-bounded $\mathcal{F}$-permitting for the identity function ($f = id$) then A is straight $\mathcal{F}$-permitting, and if A is straight $\mathcal{F}$-permitting for some $\mathcal{F}$ then A is straight multiply permitting. (Remark. These straight permitting notions were not explicitly introduced in AS2018.)

Multiple permitting: basic properties

In AS2018 we have shown the following on our formal multiple permitting notion.

- The notion is independent of the underlying enumerations: if A is f-bounded $\mathcal{F}$-permitting and $\{A_s\}_{s\geq 0}$ is any computable enumeration of A, then A is f-bounded $\mathcal{F}$-permitting via $\{A_s\}_{s\geq 0}$.
- The notion is independent of the underlying arrays: if A is multiply permitting then, for any v.s.a. $\mathcal{F}$, there is a computable function $f = f_{\mathcal{F}}$ such that A is f-bounded $\mathcal{F}$-permitting. (In contrast, the a.n.c. sets are not array invariant – though they are array invariant up to wtt-equivalence.)
- The notion is wtt-invariant (in fact closed upward under $\leq_{wtt}$): If A is multiply permitting, B is c.e., and $A \leq_{\mathrm{wtt}} B$ then B is multiply permitting too. (In contrast, the a.n.c. sets are not closed under wtt-equivalence.)
- Relation to a.n.c.: The wtt-degrees of the a.n.c. and multiply permitting degrees coincide. Moreover, any $\mathcal{F}$-a.n.c. set is straight $\mathcal{F}$-permitting, but there are multiply permitting sets which are not a.n.c.

The question whether the a.n.c. sets coincide with the straight multiply permitting sets, however, is not discussed in AS2018.

Straight multiple permitting vs. array noncomputable

THEOREM (AS). Any multiply permitting set is straight multiply permitting (but, in general, not via the same v.s.a. $\mathcal{F}$).

COROLLARY. There are straight multiply permitting sets which are array computable. In fact, for any a.n.c. set A there is a wtt-equivalent straight multiply permitting set which is not a.n.c.

So, up to wtt-equivalence, the array noncomputable sets and the (straight) multiply permitting sets coincide, but the a.n.c. sets form a proper subclass of the class of the straight multiply permitting sets which in turn coincides with the class of the multiply permitting sets.

So if we prove results on degrees, it does not matter with which notion we work. If we consider properties of sets, however, there might be a difference. By looking at the simplicity properties of the (straight) multiply permitting sets we show that such differences actually occur.

Simplicity properties of the multiply permitting sets

As mentioned before, Downey, Jockusch and Stob have completely analysed the simplicity properties of the a.n.c. sets. In particular, they have shown that

- a.n.c. sets are not dense simple (hence not hh-simple and not maximal)
- a.n.c. sets are not strongly h-simple (hence not r-maximal and not maximal)

The former extends to the multiply permitting sets (thereby giving a considerable stronger result).

THEOREM (AS2018). No multiply permitting set is dense simple.

COROLLARY. No multiply permitting set (hence no a.n.c. set) is wtt-reducible to any dense simple (hh-simple, maximal) set.

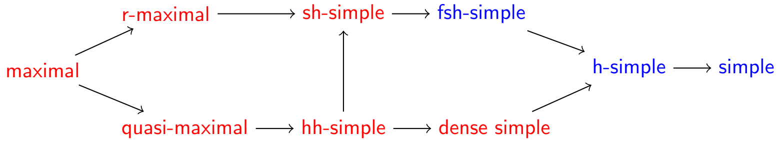
COROLLARY. Any high c.e. degree contains both, multiple permitting sets and not multiply permitting sets.

The latter, however, does not extend to the multiply permitting sets.

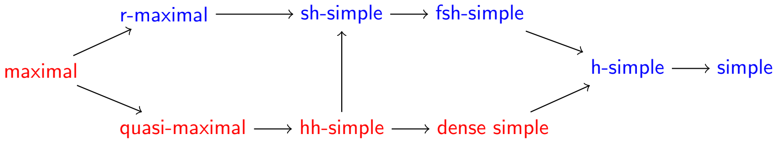
THEOREM (Monath 2018) There is a multiply permitting r-maximal set.

Note that these results completely characterize the (common) simplicity properties that a multiply permitting set may have.

Simplicity properties: a.n.c. vs. m.p. (summary)



Simplicity properties compatible (incompatible) with array noncomputability



Simplicity properties compatible (incompatible) with multiply permitting

Alternative characterizations of the a.n.c. T-degrees

DJS1990 and DJS1996 have given the following two alternative characterizations of the a.n.c. *degrees*. Both are based on an analysis of the complexity of the functions computable by sets of a.n.c. degree (with respect to growth rate and the complexity of computable approximations of these functions in terms of their mindchanges, respectively).

THEOREM (DJS1996). For a c.e. set A, the following are equivalent.

(i) $deg_T(A)$ is a.n.c.

(ii) For any function $h \leq_{wtt} \emptyset'$ there is a function $g \leq_T A$ which is not dominated by h.

(Note that, by a classical result of Martin, there is a corresponding characterization of the nonlow$_2$ degrees where $h \leq_{wtt} \emptyset'$ is replaced by $h \leq_T \emptyset'$.)

DJS1996 use this equivalence to extend the notion of an a.n.c. degree to arbitrary degrees and show that it is a very useful tool in this extended setting too.

Alternative characterizations of the a.n.c. T-degrees (ctd.)

THEOREM (DJS1990). Let f be a strictly increasing computable function and let A be a c.e. set. The following are equivalent.

(i) $deg_T(A)$ is a.n.c.

(ii) There is a set $B \leq_T A$ such that B is not f-computably approximable.

(And similarly, for functions g in place of sets B.)

Here a function g is f-computably approximable (f-c.a.) if there is a computable approximation $g(-,-)$ of g such that

$$|\{s : g(x, s+1) \neq g(x, s)\}| \leq f(x).$$

A.n.c. T-degrees vs. a.n.c. and m.p. sets

The preceding characterizations of the a.n.c. degrees proved to be very useful and led to new related notions which turned out to be as important (maybe even more important) tools for the study of the c.e. degrees and sets, and there are many applications of array noncomputability in the literature working with these characterizations.

Still, sometimes, working with these characterizations is rather tedious, and working with the multiply permitting (or a.n.c.) sets may not only simplify the arguments but at the same time may also give some stronger results.

Assume that we want to show that we can find a c.e. set A with a certain property P in any given a.n.c. degree $\mathbf{a}$ where P can be forced by a priority argument. Then a typical proof in the literature using the degree characterization of the a.n.c. degrees repeats the priority argument for constructing a P-set A and combines it with the necessary permitting (which is somewhat implicitly given by the degree characterization) to make $A \leq_T \mathbf{a}$ together with some coding to make $\mathbf{a} \leq_T A$. In particular, the proof is synthetic (constructive).
In many cases, using the same basic ideas, we can show that any m.p. set A has this property P. This is done by an analytic (modular) proof for which it suffices to consider a single requirement of the construction of a P-set in isolation (so the tedious parts of the combinatorics of the priority proof and the coding become completely superfluous and the permitting part becomes simplified). Moreover, since the class of the m.p. sets is closed upwards under wtt-reducibility in the c.e. sets we get a stronger result.

Not totally ω-c.a. degrees - a stronger multiple permitting notion

(Equivalent characterizations in terms of a.n.c. and multiply permitting sets)

A stronger variant of multiple permitting

The array noncomputable sets and the multiply permitting sets have been designed to give the required permitting in a priority construction where, in order to meet a requirement R_e, a sequence of up to $f(x)$ numbers has to be permitted where the computable function f does not depend on the requirement. In many constructions, however, there is such a dependence. So the function f has to be replaced by a sequence of computable functions f_e, hence a stronger multiple permitting is needed.

Following a suggestion of Joe Miller, Downey, Greenberg and Weber (2007) introduced the class of the not totally ω-c.a. degrees which contain the sets with this stronger permitting property. Their definition of this class is obtained by adjusting the characterization of the a.n.c. degrees in terms of the mind-change complexity of the computed functions.

DEFINITION (DGW2007). A c.e. degree $\mathbf{a}$ is not totally ω-c.a. if there is a function $g \leq_T \mathbf{a}$ that is not ω-c.a., i.e., that is not f-c.a. for any computable function f.

Some basic facts on the not totally ω-c.a. degrees

- (DGW2007) The class of the n.t.ω-c.a. degrees is properly contained in the class of the a.n.c. degrees.

- (DGW2007) The class of the n.t.ω-c.a. degrees is closed upwards in the c.e. degrees, it properly contains the class of the c.e. nonlow$_2$ degrees is properly contained in the class of the n.t.ω-c.a. degrees, ...

- (DGW2007) The class of the n.t.ω-c.a. degrees is definable: a c.e. degree **a** bounds a critical triple iff **a** is n.t.ω-c.a.

  Related to this AS and Losert (ta) have shown that the lattice $\mathcal{S}_7$ can be embedded in the c.e. degrees below a c.e. degree **a** iff **a** is n.t.ω-c.a. For comparison, the nonmodular 5-element lattice $\mathcal{N}_5$ can be embedded below any c.e. degree $\mathbf{a} > \mathbf{0}$, whereas - as shown by Downey and Greenberg (2020) - bounding the embedding of the nondistributive modular 5-element lattice $\mathcal{M}_5$ requires a still stronger permitting notion, namely a set of not totally $< \omega^\omega$-c.a. degree.

- (Barmpalias, Downey, Greenberg 2010) A c.e. degree **a** contains a c.e. set which is not wtt-reducible to any h-simple set iff **a** is n.t.ω-c.a.

Are there universally a.n.c. sets?

In case of the a.n.c. degrees we can single out the sets which give the desired form of permitting, namely the a.n.c. sets (up to wtt-equivalence) and the multiply permitting sets. Can we do the same for the stronger multiple permitting pertaining to the n.t.ω-c.a. degrees?

Since the lengths of the members in a v.s.a. $\mathcal{F}$ correspond to the number $f(x)$ of permissions needed, a set A which is $\mathcal{F}$-a.n.c. for *all* very strong arrays $\mathcal{F}$ will have the desired permitting power. Unfortunately, however, as observed by DJS1990 already, such sets do not exist.

For guaranteeing the desired uniform multiple permitting, however, sets which are universally a.n.c. in a weaker sense suffice. Such sets were introduced by AS and Losert.

Some notions of universally a.n.c. sets

DEFINITION (AS and Losert (ta)).

(i) Let $\{\mathcal{F}^e\}_{e \geq 0}$ be a standard numbering of the very strong arrays (together with the finite initial segments of such arrays). A c.e. set A is uniformly universally a.n.c. if, for any e such that $\mathcal{F}^e$ is a v.s.a., $A^{\langle e \rangle} = \{x : \langle e, x \rangle \in A\}$ is $\mathcal{F}^e$-a.n.c.

(ii) A c.e. set A is universally a.n.c. if, for any v.s.a. $\mathcal{F}$, there is a number e such that $A^{\langle e \rangle}$ is $\mathcal{F}$-a.n.c.

(iii) A c.e. set A has the uniform bounding property if there is a s.i. computable function f such that, for any v.s.a. $\mathcal{F}$ there is an $\mathcal{F}$-a.n.c. set B such that $B \leq_{f-T} A$.

THEOREM (AS and Losert (ta)). The above notions coincide up to wtt-equivalence. Moreover the c.e. degrees which contain sets with these properties are just the n.t.ω-c.a. degrees.

Moreover, obviously, the class of the sets with the uniform bounding property is closed upward under wtt-reducibility in the c.e. sets (whereas the other two notions are not wtt-invariant).

How about uniformly multiply permitting sets?

As mentioned before, if A is multiply permitting then, for any v.s.a. $\mathcal{F}$, there is a computable function f such that A is f-bounded $\mathcal{F}$-permitting. In general, however, this function f depends on the chosen array. For obtaining the desired (n.t.ω-c.a.)-permitting, we need that this bound is array-independent.

DEFINITION (AS and Losert (ta)). A c.e. set A is uniformly multiply permitting if there is a computable function f such that A is $\mathcal{F}$-permitting via f for all v.s.a.s $\mathcal{F}$.

THEOREM (AS and Losert (ta)). A c.e. set is uniformly multiply permitting iff it is wtt-equivalent to some set with the uniform bounding property. Hence a c.e. degree **a** is uniformly multiply permitting iff **a** is not totally ω-c.a. Moreover, the class of the uniformly multiply permitting sets is closed upwards under wtt-reducibility in the c.e. sets.

Working with uniformly multiply permitting sets

Just as in case of the a.n.c. degrees, working with uniformly multiply permitting sets in place of the degree characterization of the not totally ω-c.a. degrees can greatly simplify *some* of the proofs and – at the same time – strengthen the results. Again constructive proofs can be replaced by analytic ones thereby avoiding the machinery of priority arguments and coding and simplifying the permitting part.

In the following we give an example (more examples can be found in AS and Losert (ta) and AS, Losert and Monath (ta)).

Uniformly multiply permitting sets: a sample application

We give a greatly simplified proof of Barmpalias, Downey and Greenberg's result of 2010 that any totally ω-c.a. degree **a** contains a c.e. set which is not wtt-reducible to any h-simple set.

THEOREM (AS-Losert (ta)). No uniformly multiply permitting set is hypersimple.

By wtt-upward closure of the uniformly multiply permitting sets we get the desired result.

COROLLARY (AS-Losert (ta)). No uniformly multiply permitting set is wtt-reducible to any hypersimple set. (Hence any not totally ω-c.a. degree **a** contains a c.e. set which is not wtt-reducible to any h-simple set.)

Since (by Dekker) any c.e. degree contains an h-simple set, we further get a general inhomogeneity result in this setting (contrasting a homogeneity theorem of Monath for the multiply permitting sets).

COROLLARY (AS-Losert (ta)). Any not totally ω-c.a. degree **a** contains a c.e. wtt-degree which is not uniformly multiply permitting.

U.m.p. sets: a sample application (Proof of the theorem)

- Let A be h-simple and let f be a strictly increasing computable function f. It suffices to define a v.s.a. $\mathcal{F} = \{F_n\}_{n \geq 0}$ such that A is not f-bounded $\mathcal{F}$-permitting.
- Let $\hat{\mathcal{F}} = \{\hat{F}_m\}_{m \geq 0}$ be the unique v.s.a.i. such that $\hat{F}_0 = \{0\}$ and, for any $m \geq 0$, $\hat{F}_{m+1} = \hat{F}^{\ell}_{m+1} \cup \hat{F}^{r}_{m+1}$ where $|\hat{F}^{\ell}_{m+1}| = |\bigcup_{m' \leq m} \hat{F}_{m'}| + 1$ and $\max \hat{F}^{r}_{m+1} = f(\max \hat{F}^{\ell}_{m+1}) + 1$.
- Now, since A is h-simple, there are infinitely many m such that $\hat{F}_m \subseteq A$. So, since A is c.e., we get an infinite computable sequence $0 =< m_0 < m_1 < m_2 < \dots$ such that $\hat{F}_{m_n} \subseteq A$ (for all $n \geq 0$).
  Let $F_0 = \bigcup_{m \leq m_0} \hat{F}_m$ and $F_{n+1} = \bigcup_{m_n < m \leq m_{n+1}} \hat{F}_m$.
  Note, that for any n, $F_n = L_n \cup M_n \cup R_n$ such that $|M_n| > |\{x : x < \min M_n\}|$, $f(\max M_n) < \max R_n$, and $M_n \cup R_n \subseteq A$.
- It remains to show that A is not f-bounded $\mathcal{F}$-permitting. For this sake it suffices to define a partial computable function ψ such that for all $n \geq 0$,

$$(*) \quad \exists x \in F_n \, (\psi(x)\downarrow \ \& \ A_{\psi(x)} \upharpoonright f(x) + 1 = A \upharpoonright f(x) + 1).$$

  Given n, define ψ on F_n as follows: Wait for s_0 such that $M_n \cup R_n \subseteq A_{s_0}$ then we use the numbers $x \in M_n$ in order of magnitude to attack $(*)$ by letting $\psi(x)$ be the current stage where we start a new attack when A repels the previous attack. Since the only numbers $\leq f(\max M_n)$ which are not in A_{s_0} are less than $\min M_n$ and since there are more numbers in M_n than to the left of it, eventually we succeed. □

Concluding Remarks

Summary

- We looked at the a.n.c. degrees and the not totally ω-degrees, i.e., the classes of degrees which contain c.e. sets which provide multiple permittings of computably bounded finite sequences of numbers where in the first case the computable bound must not depend on requirements whereas in the second case such a dependance may exist.
- For either of these degree classes we isolated the c.e. sets which give the desired permittings, namely the multiply permitting sets and the uniformly multiply permitting sets, respectively. In the former case, a sufficient condition for providing the desired permittings were known before, namely the a.n.c. sets. In the latter case no related notions had been introduced previously.
- In either case we have shown that our formal multiple permitting notion is upward closed under wtt-reducibility in the c.e. sets.
- We have explained the usefulness and advantages of our formal multiple permitting notions, in particular for proving (and extending) results dealing with strong reducibilities or structural properties of sets involving multiple permittings.

An outlook

The work we have presented might be extended in two directions:

- The a.n.c. and the not totally ω-c.a. degrees are only the lowest two levels of the Downey-Greenberg Hierarchy (2020) of the not totally α-c.a. degrees ($\alpha < \varepsilon_0$) where the complexity of a c.e. degree **a** is measured in terms of the mind-change complexity of the computable approximations of the functions $\leq_T \mathbf{a}$.

  It seems that the question of the existence of corresponding multiply-permitting-set notions for such higher levels has not yet been explored.

- Plain permittings are provided not only by c.e. sets but also by almost-c.e. sets (i.e., left-c.e. reals). These sets (reals) play an important role in computable analysis and algorithmic randomness. So it is natural to ask whether (or how) the notion of an a.n.c. or (uniformly) permitting set can be adjusted to these larger class of sets.

  A definition of array noncomputability for the almost-c.e. sets has been proposed (and applied) by AS, Fang, Losert, Merkle and Monath, and the notion of a universally a.n.c. almost-c.e. set pertaining to the not totally-ω-c.a. degrees has been introduced (and applied) by AS, Losert and Monath (see AS, Losert, Monath [ta]).

THANK YOU!