Variations on Weihrauch degrees

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1. Thank you for the invitation to talk at this year's CiE. I am sorry I cannot attend it in person, but life just would not allow it.

Talk outline

1. Extended Weihrauch degrees
2. Variations on Weihrauch degrees
3. Relating variations with simulations

1. The title of my talk promises variations on Weihrauch degrees. What do I mean by that? We will modify the usual Weihrauch reducibility in two ways.
2. First, we extend Weihrauch degrees to a larger lattice with a better structure. The larger structures enriches the usual Weihrauch reductions and allows us to incorporate new, mathematically relevant degrees.
3. Second, we need not use the Type 2 machine model to carry out the reductions. By using other computational models, we obtain variations of Weihrauch degrees. For example, Type 1 machines yield truth-table-style reductions, and there are many others.
4. In the last part we shall see how John Longley's notion of simulations between computational models induce homomorphisms between variations of Weihrauch degrees.

Extended Weihrauch degrees

1. Let us proceed to extended Weihrauch degrees.
2. I arrived at these by interpreting instance reducibilities in the Kleene-Vesley topos, see "Instance reducibility and Weihrauch degrees" (arXiv:2106.01734). The paper also contains many technical details that we cannot attend to here.
3. However, in this talk we shall avoid the connection with constructive mathematics and realizability toposes. Instead, we shall directly generalize the ordinary Weihrauch degrees, and try to motivate our steps independently from the connection with instance reducibilities.

Notation

- Coding in $\mathbb{N}$:
  - pairs: $\langle \square, \square \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$,
  - lists: $[n_1, \ldots, n_k] = \langle n_1, \langle n_2, \langle \cdots, n_k \rangle \rangle \rangle$.
- Let $\boldsymbol{\varphi}_n$ be the n-th partial computable map.
- Let $\boldsymbol{\varphi}_n^{\mathbb{O}}$ be the n-th partial $\mathbb{O}$-computable map, $\mathbb{O} \in 2^{\mathbb{N}}$.
- Baire space: $\mathbb{B} := \mathbb{N}^{\mathbb{N}}$.
- Computable Baire space: $\mathbb{B}' := \{\alpha \in \mathbb{B} \mid \alpha \text{ computable}\}$.
- Numerals in $\mathbb{B}$: for $n \in \mathbb{N}$, let $\overline{n} := (k \mapsto n)$.
- Each $\alpha \in \mathbb{B}'$ represents a partial map $\boldsymbol{\eta}_\alpha : \mathbb{B} \rightharpoonup \mathbb{B}$.
  Write $\alpha \cdot \beta := \boldsymbol{\eta}_\alpha(\beta)$ and $(\alpha \cdot \beta){\downarrow}$ when $\alpha \cdot \beta$ is defined.

1. But first, let us review some basic notation.

(Ordinary) Weihrauch reducibility

- **(Ordinary) Weihrauch predicate**: a map $U : \mathbb{B} \to \mathscr{P}(\mathbb{B})$.
- The **support** $\|U\| := \{\alpha \in \mathbb{B} \mid U(\alpha) \neq \emptyset\}$.
- A **reduction** $U \leq_{\mathcal{W}} V$ is given by $\ell_1, \ell_2 \in \mathbb{B}'$ such that, for all $\alpha \in \|U\|$,
  - $(\ell_1 \cdot \alpha)\downarrow$, and
  - if $\beta \in V(\ell_1 \cdot \alpha)$ then $(\ell_2 \cdot \alpha \cdot \beta)\downarrow$ and $\ell_2 \cdot \alpha \cdot \beta \in U(\alpha)$.
- Let $\mathcal{W} := \mathscr{P}(\mathbb{B})^{\mathbb{B}}$.
- $(\mathcal{W}, \leq_{\mathcal{W}})$ is the preorder of Weihrauch predicates.

1. Recall the definition of a Weihrauch problem. There are several presentations, I chose here one that generalizes most directly.
2. We think of $\alpha \in \|U\|$ as an *instance* of a *problem* U. The elements of $U(\alpha)$ are the *solutions*.
3. The central notion is a Weihrauch reduction $U \leq_{\mathcal{W}} V$. It consists of two comptuable functions, represented by computable ℓ_1, ℓ_2:
   - ℓ_1 translates an U-instance α to a V-instance $\ell_1 \cdot \alpha$.
   - ℓ_2 translates a V-solution of $\ell_1 \cdot \alpha$ to a U-solution of α.

Extended Weihrauch reducibility

- **Extended Weihrauch predicate**: a map $U : \mathbb{B} \longrightarrow \mathscr{P}(\mathscr{P}(\mathbb{B}))$.
- The **support** $\|U\| := \{\alpha \in \mathbb{B} \mid U(\alpha) \neq \emptyset\}$.
- A **reduction** $U \leq_{\mathscr{W}} V$ is given by $\ell_1, \ell_2 \in \mathbb{B}'$ such that, for all $\alpha \in \|U\|$,
  - $(\ell_1 \cdot \alpha)\!\downarrow$, and
  - for every $\Theta \in U(\alpha)$ there is $\Xi \in V(\ell_1 \cdot \alpha)$ such that:
    if $\beta \in \Xi$ then $(\ell_2 \cdot \alpha \cdot \beta)\!\downarrow$ and $\ell_2 \cdot \alpha \cdot \beta \in \Theta$.
- Let $\mathscr{W} := \mathscr{P}(\mathscr{P}(\mathbb{B}))^{\mathbb{B}}$.
- $(\mathscr{W}, \leq_{\mathscr{W}})$ is the preorder of extended Weihrauch predicates.

1. We now generalize Weihrauch degrees as follows.
2. A predicate maps to the *double* powerset, so each instance $\alpha \in \|U\|$ may have many solution sets.
3. A reduction $U \leq_{\mathscr{W}} V$ has a new part, shown in color, which introduces an additional component: for every U-solution set Θ for α there must exist (non-computably) a V-solution set Ξ in the corresponding instance $\ell_1 \cdot \alpha$, such that ℓ_2 translates solutions in Ξ to solutions in Θ.
4. Note that ℓ_2 does not "know" which Θ and Ξ it is working with. The effect of this is that ℓ_2 must translate solutions *uniformly* in Ξ, but we may help it by *non-uniformly* selecting a suitable Ξ.

The structure of $(\mathscr{W}, \leq_{\mathscr{W}})$

- $\mathscr{W}$ is a Heyting algebra with implication

$$
\begin{aligned}
(U \Rightarrow V)(\alpha) :=& \\
&\{\Psi \in \mathscr{P}(\mathbb{B}) \mid (\forall \beta \in \|U\|.\, \mathtt{fst} \cdot (\alpha \cdot \beta) \in \|V\|) \land \\
&\qquad \forall \Theta \in U(\beta).\, \exists \Xi \in V(\mathtt{fst} \cdot (\alpha \cdot \beta)). \\
&\qquad \forall \gamma \in \Xi.\, \mathsf{snd} \cdot (\alpha \cdot \beta) \cdot \gamma \in \Theta \oplus \Psi\},
\end{aligned}
$$

  where
$$\Theta \oplus \Psi := \{0\alpha \mid \alpha \in \Theta\} \cup \{1\beta \mid \beta \in \Psi\}.$$

- $\mathscr{W}$ is computably complete (in a suitable sense).

1. The extended Weihrauch predicates have a much better structure than the ordinary ones.
2. Internally to the Kleene-Vesley realizability topos they even form a complete Heyting lattice.

Embedding $\mathscr{w} \longrightarrow \mathscr{W}$

For an ordinary Weihrauch predicate $U : \mathbb{B} \longrightarrow \mathscr{P}(\mathbb{B})$, define

$$\begin{aligned}
\overline{U} &: \mathbb{B} \longrightarrow \mathscr{P}(\mathscr{P}(\mathbb{B})), \\
\widehat{U} &: \alpha \mapsto \begin{cases} \{U(\alpha)\} & \text{if } \alpha \in \|U\|, \\ \emptyset & \text{otherwise.} \end{cases}
\end{aligned}$$

The map $U \mapsto \widehat{U}$ is an embedding $\mathscr{w} \longrightarrow \mathscr{W}$:

$$U \leq_{\mathscr{w}} W \iff \overline{U} \leq_{\mathscr{W}} \overline{V}.$$

1. We may embed ordinary degrees into extended ones. The embedding is fairly simple: an ordinary degree only has one solution set.

Examples: non-ordinary extended predicates

- The largest degree $\top(\alpha) := \{\emptyset\}$
- Church's thesis $\mathsf{CT}(\alpha) = \{\{\overline{n} \mid n \in \mathbb{N} \wedge \varphi_n = \alpha\}\}$
- Weak Excluded middle $\mathsf{WLEM}(\alpha) = \{\{\overline{0}\}, \{\overline{1}\}\}$.
- Excluded middle:

$$\begin{aligned}\mathsf{LEM}(\alpha) = &\{\{0\beta \mid \beta \in \mathbb{B}\}\} \cup \\ &\quad \{\{1\gamma \mid \gamma \in S\} \mid S \in \mathscr{P}(\mathbb{B}) \wedge S \neq \emptyset\}.\end{aligned}$$

1. Here are some extended Weihrauch predicates that are not ordinary.
2. The top degree has an empty solution set. (An ordinary Weihrauch degree cannot.)
3. Church's thesis is interesting because it has single solution sets, but some of them are empty. (Whereas in an ordinary degree they must always be non-empty.)
4. The weak excluded middle states that $\neg p \vee \neg\neg p$ for all truth values p. The corresponding degree demonstrates how multiple solution sets are used: in a reduction of U to WLEM we may select classically, i.e., without any computable witness, either the solution set $\{\overline{0}\}$ or $\{\overline{1}\}$.
5. Full excluded middle is more complicated because it also carries solution sets witnessing the fact that in $\neg p \vee p$ the right disjunct p holds.

Variations of Weihrauch degrees

1. Let's proceed to a generalization of Weihrauch degrees to other models of computation.

Partial combinatory algebra (pca)

A **pca** is a set $\mathbb{E}$ with a partial operation $\cdot : \mathbb{E} \times \mathbb{E} \rightharpoonup \mathbb{E}$ such that there are $\mathsf{k}, \mathsf{s} \in \mathbb{E}$ satisfying, for all $a, b, c \in \mathbb{E}$,

$$
\begin{aligned}
&(\mathsf{k}\,a)\downarrow, && \mathsf{k}\,a\;b = a, \\
&(\mathsf{s}\,a\,b)\downarrow, && \mathsf{s}\,a\,b\,c \simeq (a\,c)\,(b\,c).
\end{aligned}
$$

A **sub-pca** is a set $\mathbb{E}' \subseteq \mathbb{E}$ closed under application and $\mathsf{k}, \mathsf{s} \in \mathbb{E}'$.

We always consider a pca $\mathbb{E}$ with a chosen sub-pca $\mathbb{E}'$.

1. Let us proceed with variations of Weihrauch degrees. Instead of using the Baire space, we shall allow any model of computation.
2. By "model of computation" we mean a partial combinatory algebra.

Examples of pcas with sub-pcas

- Turing model:
  $\mathbb{T}' = \mathbb{T} = \mathbb{N}$ and $m \cdot n = \boldsymbol{\varphi}_m(n)$.
- Oracle model:
  - Fix an oracle $\mathbb{O} \in 2^{\mathbb{N}}$,
  - $\mathbb{T}_{\mathbb{O}}{}' = \mathbb{T}_{\mathbb{O}} = \mathbb{N}$ and $m \cdot n = \boldsymbol{\varphi}_m^{\mathbb{O}}(n)$.
- Kleene-Vesley model:
  $\mathbb{B} = \mathbb{N}^{\mathbb{N}}$ and $\mathbb{B}' = \{\alpha \in \mathbb{B} \mid \alpha \text{ computable}\}$.
- Scott's graph model:
  $\mathbb{P} = \mathscr{P}(\mathbb{N})$ and $\mathbb{P}' = \{S \in \mathscr{P}(\mathbb{N}) \mid S \text{ is c.e.}\}$.

1. In Scott's model application is defined as
   $S \cdot T = \{n \in \mathbb{N} \mid \exists m_1, \ldots, m_k \in T.\, \langle [m_1, \ldots, m_k], n\rangle \in S\}$, where $\langle \square, \square \rangle$ is a pairing function and $[\square]$ codes finite sequences.

Generalized Weihrauch reducibility

- Let $\mathbb{E}$ be a pca with a sub-pca $\mathbb{E}'$.
- **Extended Weihrauch predicate**: a map $U : \mathbb{E} \to \mathscr{P}(\mathscr{P}(\mathbb{E}))$.
- The **support** $\|U\| := \{a \in \mathbb{E} \mid U(a) \neq \emptyset\}$.
- A **reduction** $U \leq_{\mathscr{W}} V$ is given by $\ell_1, \ell_2 \in \mathbb{E}'$ such that, for all $a \in \|U\|$,
  - $(\ell_1 \cdot a)\downarrow$, and
  - for every $\Theta \in U(a)$ there is $\Xi \in V(\ell_1 \cdot a)$ such that: if $b \in \Xi$ then $(\ell_2 \cdot a \cdot b)\downarrow$ and $\ell_2 \cdot a \cdot b \in \Theta$.
- Let $\mathscr{W}_{\mathbb{E}} := \mathscr{P}(\mathscr{P}(\mathbb{E}))^{\mathbb{E}}$.
- $(\mathscr{W}_{\mathbb{E}}, \leq_{\mathscr{W}})$ is the preorder of extended Weihrauch predicates.
- Ordinary Weihrauch predicates $\mathscr{w}_{\mathbb{E}}$ are defined analogously.

1. The definition of Weihrauch predicates translates directly to any pca $\mathbb{E}$ with a sub-pca $\mathbb{E}'$.

$\mathscr{W}_{\mathbb{T}}$ – Turing machine model

The **membership problem** associated with $A \subseteq \mathbb{N}$ is the ordinary Weihrauch predicate $\chi_A \in \mathscr{W}_{\mathbb{T}}$,

$$\chi_A : \mathbb{N} \longrightarrow \mathscr{P}(\mathbb{N})$$
$$\chi_A : n \mapsto \begin{cases} \{1\} & \text{if } n \in A, \\ \{0\} & \text{if } n \notin A. \end{cases}$$

For $A, B \subseteq \mathbb{N}$ we have

$$A \leq_{\mathsf{T}} B \iff \chi_A \leq_w \chi_B^{(\mathbb{N})},$$
$$A \leq_{\mathsf{wtt}} B \iff \chi_A \leq_w \bigsqcup_{k \in \mathbb{N}} \chi_B^{(k)}.$$

where $\leq_{\mathbb{T}}$ and $\leq_{\mathsf{wtt}}$ are respectively Turing and weak-truth-table reducibilities.

1. When we specialize to Turing machine model we get a notion of reducibility that encomapsses both, Turing and weak-truth-table reducibilities.
2. In general, the predicate $V^{(\mathbb{N})}$ is $\mathbb{N}$-many instances of V packed together, so that $U \leq_w V^{(\mathbb{N})}$ corresponds to a reduction of one instance of U to an arbitrary number of instances of V.
3. The predicate $\bigsqcup_{k \in \mathbb{N}} V^{(k)}$ is like a disjoint sum of predicates $V^{(k)}$, each packing k-many instances of U. A reduction to it must compute a specific k, hence the relationship with truth-table reducibility.

$\mathscr{W}_{\mathbb{P}}$ – Scott's graph model

For $A \subseteq \mathbb{N}$ define the Weihrauch predicate $E_A \in \mathscr{W}_{\mathbb{P}}$,

$$E_A : \mathbb{P} \longrightarrow \mathscr{P}(\mathbb{P})$$
$$E_A : S \mapsto \{A\}.$$

For $A, B \subseteq \mathbb{N}$ we have

$$A \leq_{\mathsf{e}} B \Leftrightarrow E_A \leq_{\mathscr{W}} E_B$$

where $\leq_{\mathsf{e}}$ is *enumeration reducibility*.

1. In Scott's graph model we obtain a notion of reducibility that is akin to enumeration reducibility.

Relating variations with simulations

1. Just collecting various notions of Weihrauch reducibility is a bit like stamp-colecting. We really should also study how they are related.
2. For this purpose, we shall use John Longley's notion of simulations. An older name for these is applicative morphisms.

Simulations between pcas

A **simulation** $f : \mathbb{E} \longrightarrow \mathbb{F}$ is a map $f : \mathbb{E} \longrightarrow \mathscr{P}(\mathbb{F})$ such that

- $f(a) \neq \emptyset$ for all $a \in \mathbb{E}$,
- $f(a) \subseteq \mathbb{F}'$ for all $a \in \mathbb{E}'$,
- there is $r \in \mathbb{F}'$ such that:
  for all $a, b \in \mathbb{E}$, $a' \in f(a)$ and $b' \in f(b)$,
  if $(a\, b){\downarrow}$ then $(r\, a'\, b'){\downarrow}$ and $r\, a'\, b' \in f(a \cdot b)$.

Simulations form a category:

- identity simulation: $\mathrm{id}_{\mathbb{E}} : \mathbb{E} \longrightarrow \mathbb{E}$ is $\mathrm{id}_{\mathbb{E}}(a) := \{a\}$,
- $f : \mathbb{E} \longrightarrow \mathbb{F}$ and $g : \mathbb{F} \longrightarrow \mathbb{G}$ compose to $g \circ f : \mathbb{E} \longrightarrow \mathbb{G}$,
  $(g \circ f)(a) := \bigcup_{b \in f(a)} g(b)$.

1. Here is the basic definition. A simulation may be *multi-valued*, that is, we can simulate a given element in $\mathbb{E}$ with sevral elements of $\mathbb{F}$.
2. The idea is to *simulate* one PCA inside another. The realizer r does precisely that.

Adjunctions, inclusions, retractions, equivalences

Given simulations $f_1, f_2 : \mathbb{E} \longrightarrow \mathbb{F}$, define $f_1 \preceq f_2$ to mean

$$\exists q \in \mathbb{F}'. \forall a \in \mathbb{E}. \forall a' \in f_1(a). (q\, a')\!\downarrow \wedge\, q\, a' \in f_2(a).$$

Write $f_1 \sim f_2$ when $f_1 \preceq f_2$ and $f_2 \preceq f_1$.

Simulations $f : \mathbb{E} \longrightarrow \mathbb{F}$ and $g : \mathbb{F} \longrightarrow \mathbb{E}$ form:

- an **adjunction** $g \dashv f$ when $\mathrm{id}_{\mathbb{F}} \preceq f \circ g$ and $g \circ f \preceq \mathrm{id}_{\mathbb{E}}$,
- **inclusion** when $g \dashv f$ and $g \circ f \sim \mathrm{id}_{\mathbb{E}}$.
- **retraction** when $g \dashv f$ and $f \circ g \sim \mathrm{id}_{\mathbb{F}}$,
- **equivalence** when $f \circ g \sim \mathrm{id}_{\mathbb{F}}$ and $g \circ f \sim \mathrm{id}_{\mathbb{E}}$.

1. John Longley identified particularly well-behaved combinations of simulations. They are relevant to our topic as well.

Examples: Turing & oracle models

$$h : \mathbb{T} \to \mathbb{T}_{\mathcal{O}} \qquad\qquad k : \mathbb{T}_{\mathcal{O}} \to \mathbb{T}$$

$$h : n \mapsto \{n\} \qquad\qquad k : m \mapsto \{n \in \mathbb{N} \mid \varphi_n^{\mathcal{O}}(0) = m\}$$

We have $\mathrm{id}_{\mathbb{T}} \preceq k \circ h$ and $h \circ k \sim \mathrm{id}_{\mathbb{T}_{\mathcal{O}}}$, an inclusion of $\mathbb{T}_{\mathcal{O}}$ into $\mathbb{T}$.

1. It is intuitively clear that ordinary Turing machines can be simulated by machines with an oracle $\mathcal{O}$.
2. It is less clear that the opposite direction works as well. To understand the map k, imagine that m is some number that was computed with the aid of an oracle $\mathcal{O}$.
   We cannot compute m without an oracle, so we represent it by (codes of) a machine that *would* compute it, if we had the oracle.
3. The map k is a simulation because we can construct $\mathcal{O}$-machines without access to $\mathcal{O}$.

Example: Turing & computable Kleene-Vesley models

$$i : \mathbb{T} \longrightarrow \mathbb{B}' \qquad j : \mathbb{B}' \longrightarrow \mathbb{T}$$

$$i : n \longmapsto \{\overline{n}\} \qquad j : \alpha \longmapsto \{k \in \mathbb{N} \mid \boldsymbol{\varphi}_k = \alpha\}$$

We have $i \circ j \preceq \mathrm{id}_{\mathbb{B}'}$ and $j \circ i \sim \mathrm{id}_{\mathbb{T}}$, a retraction of $\mathbb{T}$ into $\mathbb{B}'$.

1. Let us think why $i \circ j \prec \mathrm{id}_{\mathbb{B}'}$ in this example. Given $\alpha \in \mathbb{B}'$, $\beta \in i(j(\alpha)$ is of the form $\beta = \overline{n}$, i.e., a code of a machine that computes α. From such n we can reconstruct α easily.
2. But the other direction $\mathrm{id}_{\mathbb{B}'} \prec i \circ j$ does not hold, for it would require us to compute from $\alpha \in \mathbb{B}'$, the code n of a machine that computes α.

Example: Kleene-Vesley & Scott's graph models

$$f : \mathbb{B} \longrightarrow \mathscr{P}(\mathbb{P})$$
$$f : \alpha \mapsto \{\{[\alpha_0, \ldots, \alpha_{n-1}] \in \mathbb{N} \mid n \in \mathbb{N}\}\}$$

$$g : \mathbb{P} \longrightarrow \mathscr{P}(\mathbb{B})$$
$$g : S \mapsto \{\alpha \in \mathbb{B} \mid S = \{n \in \mathbb{N} \mid \exists k.\, \alpha_k = n + 1\}\}$$

We have $f \circ g \preceq \mathrm{id}_{\mathbb{P}}$ and $g \circ f \sim \mathrm{id}_{\mathbb{B}}$, a retraction of $\mathbb{P}$ onto $\mathbb{B}$.

1. One last example: there is a retarction from Scott's graph model onto the Kleene-Vesley model.

$f : \mathbb{E} \longrightarrow \mathbb{F}$ induces $f_* : \mathscr{W}_\mathbb{E} \longrightarrow \mathscr{W}_\mathbb{F}$

- A simulation $f : \mathbb{E} \longrightarrow \mathbb{F}$ induces $f_* : \mathscr{W}_\mathbb{E} \longrightarrow \mathscr{W}_\mathbb{F}$ taking $U : \mathbb{E} \longrightarrow \mathscr{P}(\mathscr{P}(\mathbb{E}))$ to

$$
\begin{aligned}
&f_*(U) : \mathbb{F} \longrightarrow \mathscr{P}(\mathscr{P}(\mathbb{F})) \\
&f_*(U) : c \mapsto \{\textstyle\bigcup_{b \in \Theta} f(b) \mid \exists a \in \mathbb{E}.\, c \in f(a) \wedge \Theta \in U(a)\},
\end{aligned}
$$

- The map f_* is $\leq_\mathscr{W}$-monotone.
- f_* restricts to $f_* : \mathscr{w}_\mathbb{E} \longrightarrow \mathscr{w}_\mathbb{F}$ when f is *discrete*:

$$
\forall c \in \mathbb{F}.\, \forall a, b \in \mathbb{E}.\, c \in f(a) \wedge c \in f(b) \Rightarrow a = b.
$$

1. A simulation induces a homomorphism between the corresponding preorders of extended Weihrauch predicates.
2. When the simulation is discrete, the induced homomorphism restricts to ordinary degrees.

Adjunctions, inclusions, retractions, equivalences

Consider simulations

$$f : \mathbb{E} \to \mathbb{F} \quad \text{and} \quad g : \mathbb{F} \to \mathbb{E}$$

and induced homomorphisms:

$$f_* : \mathscr{W}_{\mathbb{E}} \to \mathscr{W}_{\mathbb{F}} \quad \text{and} \quad g_* : \mathscr{W}_{\mathbb{F}} \to \mathscr{W}_{\mathbb{E}}$$

We have:

- if $g \dashv f$ then g_* and f_* are lattice homomorphisms.
- if $g \dashv f$ then $g_* \dashv f_*$ is a Galois connection.
- if $g \dashv f$ is an inclusion then $g_*(f_*(U)) \equiv_{\mathscr{W}_{\mathbb{E}}} U$.
- if $g \dashv f$ is a retraction then $f_*(g_*(V)) \equiv_{\mathscr{W}_{\mathbb{F}}} V$.
- if $g \dashv f$ is an equivalence then $\mathscr{W}_{\mathbb{E}} \simeq \mathscr{W}_{\mathbb{F}}$ and $w_{\mathbb{E}} \simeq w_{\mathbb{F}}$.

1. The adjunctions, inclusions, retractions and equivalences between simulations carry over to homomorphisms of extended Weihrauch predicates.
2. Moreover, adjoint simulations preserve finite infinima and suprema.

Questions & ideas

1. Can we do something useful with Heyting implication in $\mathscr{W}$?
2. Should we study $\mathscr{W}_{\mathbb{P}}$ instead of $\mathscr{W}_{\mathbb{B}}$? It is larger.
3. There are many more models to consider, e.g.,
   - $\mathscr{W}_{\Lambda}$ – the λ-calculus model,
   - $\mathscr{W}_{\mathbb{S}}$ – van Oosten's sequential functionals $\mathbb{S}$,
   - $\mathscr{W}_{\mathbb{J}}$ – Joel Hamkin's infinite-time Turing machines $\mathbb{J}$.
4. What about Weihrauch degrees in sheaf toposes?

1. Thank you for your attention.