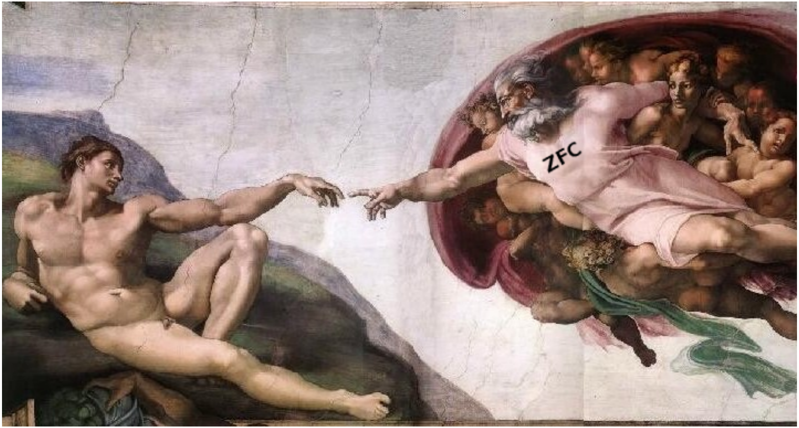
Part I : Ramsey theory computes through sparsity

Ludovic LEVY PATEY


ZFC

Theorem

$$T$$

Axioms Theorem

$$A_1, \ldots, A_n \Rightarrow T$$

Axioms Theorem

$$A_1, \ldots, A_n \Leftarrow T$$

Reverse mathematics

Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

Π^1_1CA

↓

ATR

↓

ACA

↓

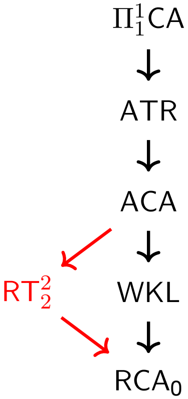
WKL

↓

RCA_0

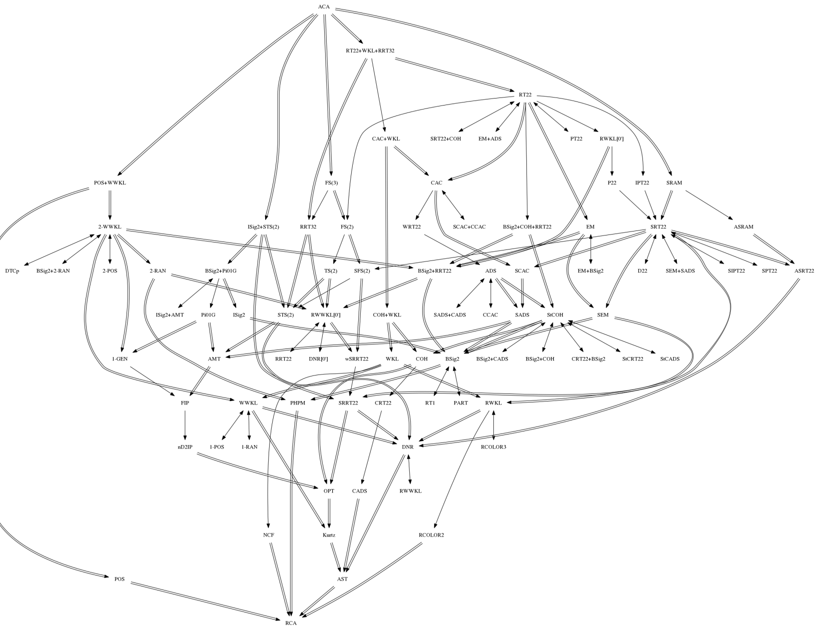
Reverse mathematics

Mathematics are computationally very structured



Almost every theorem is empirically equivalent to one among five big subsystems.

Except for Ramsey's theory...


ACA
RT22+WKL+RRT32
RT22
CAC+WKL
SRT22+COH
EM+ADS
PT22
RWKL[0']
POS+WWKL
FS(3)
CAC
P22
IPT22
SRAM
2-WWKL
ISig2+STS(2)
RRT32
FS(2)
WRT22
SCAC+CCAC
BSig2+COH+RRT22
EM
SRT22
ASRAM
DTCp
BSig2+2-RAN
2-POS
2-RAN
BSig2+Pi01G
TS(2)
SFS(2)
BSig2+RRT22
ADS
SCAC
EM+BSig2
D22
SEM+SADS
SIPT22
SPT22
ASRT22
ISig2+AMT
Pi01G
ISig2
STS(2)
RWWKL[0']
COH+WKL
SADS+CADS
CCAC
SADS
StCOH
SEM
1-GEN
AMT
RRT22
DNR[0']
wSRRT22
WKL
COH
BSig2
BSig2+CADS
BSig2+COH
CRT22+BSig2
StCRT22
StCADS
FIP
WWKL
PHPM
SRRT22
CRT22
RT1
PART
RWKL
nD2IP
1-POS
1-RAN
DNR
RCOLOR3
OPT
CADS
RWWKL
NCF
Kurtz
RCOLOR2
POS
AST
RCA

Ramsey theory

The subject

computes

The framework

through sparsity

The results

Ramsey theory

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The results

A set $A \subseteq \mathbb{N}$ is computable if there is a computer program which, on input n, decides whether $n \in A$ or not.

A set $A \subseteq \mathbb{N}$ is computable in B if there is a computer program in an language augmented with the characteristic function of B which, on input n, decides whether $n \in A$ or not.

$$A \leq_T B$$

A is computable in B

$$\Phi_e(x)\downarrow$$

The e-th program halts on input x.

$$\Phi_e(x)[t]\downarrow$$

The e-th program halts on input x
in less than t steps.

$$\Phi_e^A(x)\downarrow$$

The e-th program with oracle A halts on input x.

$$\Phi_e^A(x)[t]\downarrow$$

The e-th program with oracle A halts on input x in less than t steps.

Ramsey theory

The subject

Overall, Ramsey's theory seeks to understand the inherent structure and order that can arise within large ~~finite~~ sets by investigating the existence of specific patterns, colorings, or configurations. — ChatGPT

Ramsey's theorem

$[X]^n$ is the set of unordered n-tuples of elements of X

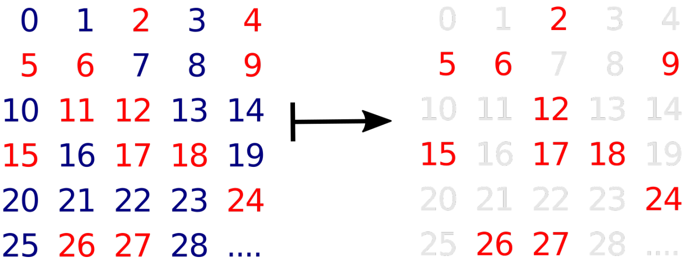
A k-coloring of $[X]^n$ is a map $f : [X]^n \to k$

A set $H \subseteq X$ is homogeneous for f if $|f([H]^n)| = 1$.

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.

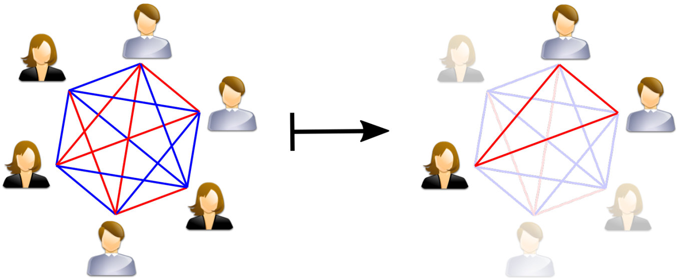
Pigeonhole principle

RT^1_k Every k-partition of $\mathbb{N}$ admits an infinite subset of a part.



Ramsey's theorem for pairs

RT^2_k Every k-coloring of the infinite clique admits an infinite monochromatic subclique.



Let $\mathbb{A}$ be a countable structure and $\mathbb{F}$ be a finite structure.
Let $[\mathbb{A}]^{\mathbb{F}}$ be the collection of sub-copies of $\mathbb{F}$ in $\mathbb{A}$.

Question

For every coloring f $: [\mathbb{A}]^{\mathbb{F}} \to$ *k, is there a sub-copy* $\mathbb{B}$ *of* $\mathbb{A}$ *such that* $[\mathbb{B}]^{\mathbb{F}}$ *is monochromatic?*

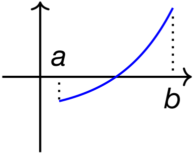
Case study: $\mathbb{A} = (\mathbb{Q}, <)$

computes

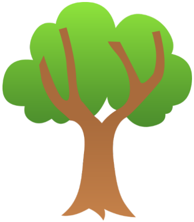
The framework

Consider mathematical problems

Intermediate value theorem
For every continuous function f over an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$, there is a real $x \in [a, b]$ such that $f(x) = 0$.



König's lemma
Every infinite, finitely branching tree admits an infinite path.



Fix a problem P.

Computable encodability

A set S is computably P-encodable if there is a computable instance of P such that every solution computes S.

Encodability

A set S is P-encodable if there is an instance of P such that every solution computes S.

Computable encodability

Thm (Jockusch and Soare)

Only computable sets are computably encodable by WKL

This is the cone avoidance Π^0_1 basis theorem

Encodability

Thm

Every set is encodable by WKL

Given a set *A*, consider the tree $T = \{\sigma \in 2^{<\mathbb{N}} : \sigma \prec A\}$

Encodability vs Domination

Encodability

A set S is P-encodable if there is an instance of P such that every solution computes S

Domination

A function f is P-dominated if there is an instance of P such that every solution computes a function dominating f.

Encodability vs Domination

The P-encodable sets
are the computable ones.

$$\neq$$

The P-dominated functions
are the computably dominated ones.

Encodability vs Domination

(relativized version)

For every Z, the P-encodable sets relative to Z
are the Z-computable ones.

$$\equiv$$

For every Z, the P-dominated functions relative to Z
are the Z-computably dominated ones.

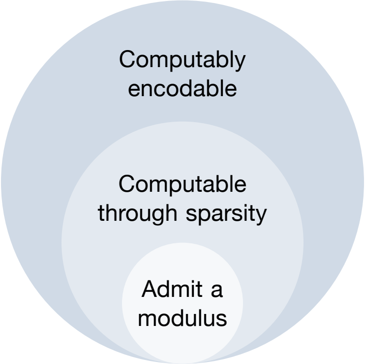
through sparsity

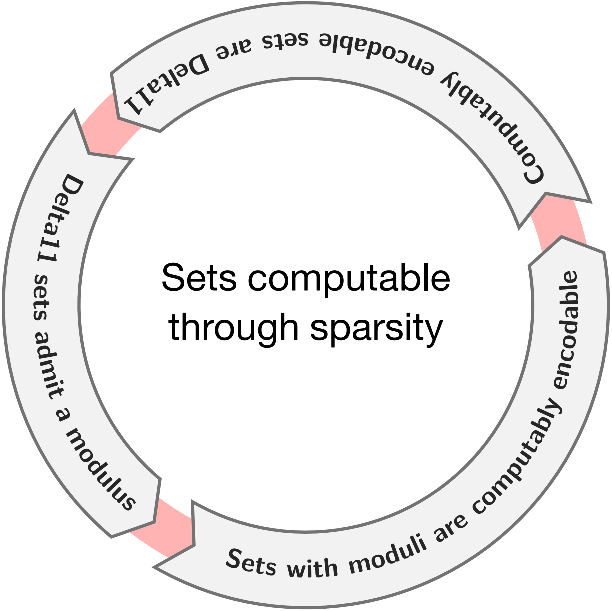
The results

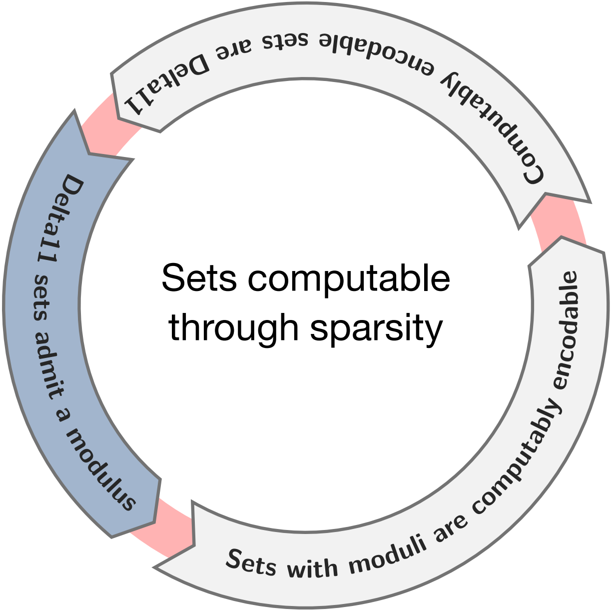
A function $f : \mathbb{N} \to \mathbb{N}$ is a modulus for a set $A \subseteq \mathbb{N}$ if every function dominating f computes A.

The principal function of an infinite set $A \subseteq \mathbb{N}$ is the function $p_A : \mathbb{N} \to \mathbb{N}$ which to n associates the nth element of A.

A set A is computably encodable if for every infinite set X, there is an infinite subset $Y \subseteq X$ computing A.




Sets computable
through sparsity
Computably encodable sets are Delta11
Delta11 sets admit a modulus
Sets with moduli are computably encodable



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What sets admit a modulus?

$\emptyset'$ admits a modulus

$$f(e) = \begin{cases} \mu_t[\Phi_e(e)[t]\downarrow] & \text{if it exists} \\ 0 & \text{otherwise} \end{cases}$$

Every function dominating f computes the halting set.

$\emptyset''$ admits a modulus

$$f(e) = \begin{cases} \mu_t[\Phi_e(e)[t]\downarrow] & \text{if it exists} \\ 0 & \text{otherwise} \end{cases}$$

$$g(e) = \begin{cases} \mu_t[\Phi_e^{\emptyset'}(e)[t]\downarrow] & \text{if it exists} \\ 0 & \text{otherwise} \end{cases}$$

Every function dominating $x \mapsto \max(f(x), g(x))$ computes the halting set of the halting set.

Arithmetic hierarchy

$\Sigma_n^0 \quad A = \{y : \exists x_1 \forall x_2 \ldots Q x_n\, R(y, x_1, \ldots, x_n)\}$

$\Pi_n^0 \quad A = \{y : \forall x_1 \exists x_2 \ldots Q x_n\, R(y, x_1, \ldots, x_n)\}$

where R is a computable predicate.

A set is Δ_n^0 if it is Σ_n^0 and Π_n^0.

Computability $\equiv$ Definability

Thm (Post)

A set is c.e. iff it is Σ^0_1 and computable iff it is Δ^0_1.

Thm (Post)

A set is $\emptyset^{(n)}$-c.e. iff it is Σ^0_{n+1} and $\emptyset^{(n)}$-computable iff it is Δ^0_{n+1}.

Thm

All the arithmetic sets admit a modulus.

$\emptyset^{(\omega)}$ admits a modulus

$$\emptyset^{(\omega)} = \bigoplus_n \emptyset^{(n)} = \left\{ \langle n, x \rangle : x \in \emptyset^{(n)} \right\}$$

Analytic hierarchy

$\Sigma^1_n \quad A = \{y : \exists X_1 \forall X_2 \ldots Q X_n \, R(y, X_1, \ldots, X_n)\}$

$\Pi^1_n \quad A = \{y : \forall X_1 \exists X_2 \ldots Q X_n \, R(y, X_1, \ldots, X_n)\}$

where R is an arithmetic predicate.

A set is Δ^1_n if it is Σ^1_n and Π^1_n.

Klenee's normal form

$$\Sigma_n^1 \quad A = \{y : \exists X_1 \forall X_2 \ldots Q X_n \, R(y, X_1, \ldots, X_n)\}$$

$$\Pi_n^1 \quad A = \{y : \forall X_1 \exists X_2 \ldots Q X_n \, R(y, X_1, \ldots, X_n)\}$$

where R is $\begin{array}{ll} \Sigma_1^0 & \text{if } Q = \forall \\ \Pi_1^0 & \text{if } Q = \exists \end{array}$

Lem (Folklore)

For every Π^1_1 set $A \subseteq \mathbb{N}$, there is a function $f : \mathbb{N} \to \mathbb{N}$ such that A is c.e. in any function dominating f.

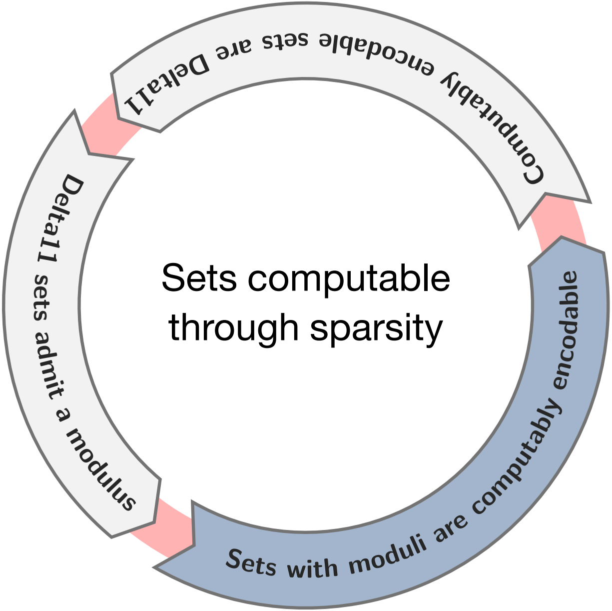
- $A = \{n \in \mathbb{N} : T_n \text{ is a well-founded tree } \}$
- Pick f such that if T_n is infinite, then $T_n \cap f_n^{<\omega}$ is infinite
- Given g dominating f, $A = \{n \in \mathbb{N} : T_n \cap g_n^{<\omega} \text{ is finite } \}$

$$\text{where given } f,\ f_n(x) = \begin{cases} f(n) & \text{if } x < n \\ f(x) & \text{otherwise} \end{cases}$$

Thm (Solovay)

All the Δ^1_1 sets admit a modulus.

- Suppose A and $\overline{A}$ are Π^1_1
- Let f, g be their c.e. moduli
- $x \mapsto \max(f(x), g(x))$ is a modulus for A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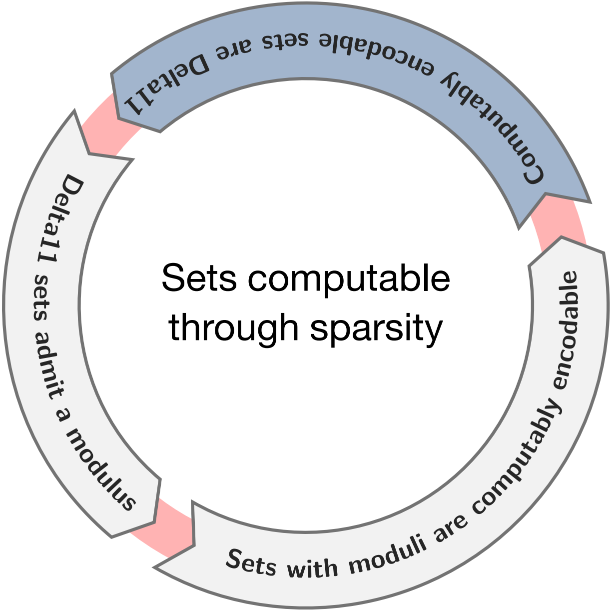
A set A is computably encodable if for every infinite set X, there is an infinite subset $Y \subseteq X$ computing A.

Thm (Folklore)

If A admits a modulus, then A is computably encodable.

Recall that p_Y is the principal function of Y.

- Let f be a modulus for A
- Given X, pick $Y \subseteq X$ be such that $p_Y \geq f$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Thm (Solovay)

If A is computably encodable, then A is Δ^1_1.

By Mathias forcing, using Galvin-Prikry's theorem

Thm (Galvin-Prikry)

For every Borel set $\mathcal{S} \subseteq [\mathbb{N}]^\omega$, there is a $B \in [\mathbb{N}]^\omega$ such that $[B]^\omega \subseteq \mathcal{S}$ or $[B]^\omega \cap \mathcal{S} = \emptyset$.

Mathias condition

$$(F, X)$$

Initial segment — Reservoir

F is finite, X is infinite,
$\max F < \min X$

Mathias extension

$$(E, Y) \leq (F, X)$$

$$F \subseteq E, Y \subseteq X, E \setminus F \subseteq X$$

Cylinder

$$[F, X] = \{G : F \subseteq G \subseteq F \cup X\}$$

Lem

Given (F, X), Φ_e and $A \notin \Delta^1_1$, there is some $Y \in [X]^\omega$ such that $\Phi_e^G \neq A$ for every $G \in [F, Y]$

- Let $\mathcal{S} = \{G \in [X]^\omega : \Phi_e^{F \cup G} = A\}$
- By Galvin-Prikry's theorem, there is $Y \in [X]^\omega$ such that

$$[Y]^\omega \subseteq \mathcal{S} \text{ or } [Y]^\omega \cap \mathcal{S} = \emptyset$$

- Assume the first case holds. Then

$$A = \{n : \forall Z \in [Y]^\omega : \Phi_e^Z(n) \downarrow = 1\}$$

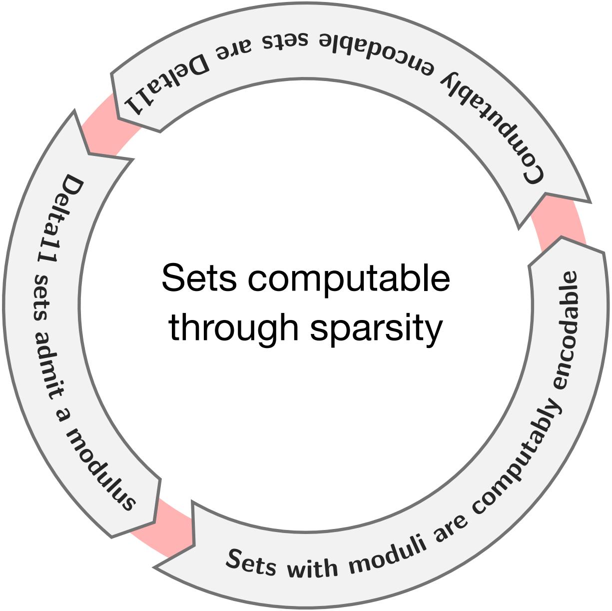
$$\overline{A} = \{n : \forall Z \in [Y]^\omega : \Phi_e^Z(n) \downarrow = 0\}$$

- Then A is Δ^1_1, contradiction.

Lem

Given (F, X), Φ_e and $A \notin \Delta^1_1$, there is some $Y \in [X]^\omega$ such that $\Phi_e^H \neq A$ for every $G \in [F, Y]$ and $H \in [G]^\omega$.

- Let $\{F_1, \ldots F_k\} = [F]^{<\omega}$
- Let Γ_i be the functional $Z \mapsto \Phi_e^{F_i \cup Z}$
- Apply successively the previous lemma to $\Gamma_1, \ldots, \Gamma_k$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Conclusion

We consider theorems as mathematical problems

A problem encodes a set if there is an instance, all of whose solutions compute the set

The Δ^1_1 sets are robust, and computable by sparsity

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