Part III : Devlin's theorem computes through sparsity

Ludovic LEVY PATEY

Devlin's theorem

Ramsey's theorem

$[X]^n$ is the set of unordered n-tuples of elements of X

A k-coloring of $[X]^n$ is a map $f : [X]^n \to k$

A set $H \subseteq X$ is homogeneous for f if $|f([H]^n)| = 1$.

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.

$(\mathbb{N}, <)$ vs $(\mathbb{Q}, <)$

Does every *k*-coloring of $[\mathbb{Q}]^n$
admit a homogeneous subcopy?

$n = 1$

Every k-coloring of $\mathbb{Q}$ admits
a homogeneous subcopy

Fix a 2-coloring of $\mathbb{Q}$

- Either one full interval has color blue
- Or the elements of color red are dense

$n = 2$

Thm (Galvin)

There is a 2-coloring of $[\mathbb{Q}]^2$ with no homogeneous subcopy

Fix an enumeration of $\mathbb{Q}$: $q_0, q_1, q_2, \ldots$

$$f(\{q_i, q_j\}) = 1 \text{ iff } q_i <_{\mathbb{Q}} q_j \Leftrightarrow i <_{\mathbb{N}} j$$

$n = 2$

Every k-coloring of $[\mathbb{Q}]^2$ admits
a subcopy with at most 2 colors

Thm (Devlin)

For every n, there is some ℓ such that for every k, every k-coloring of $[\mathbb{Q}]^n$ admits a subcopy of $\mathbb{Q}$ with at most ℓ colors

Dense linear order without endpoints

Linear order $\mathcal{L} = (L, <)$ such that for every $x, y \in \mathcal{L}$ with $x < y$, there are some $a, b, c \in L$ such that

$$a < x < b < y < c$$

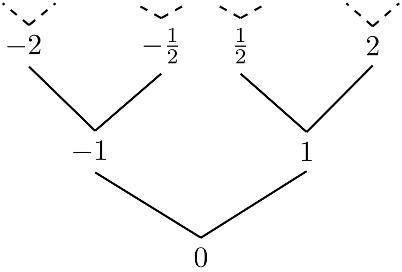
Lem

DLO are computably categorical

$$\sigma <_{\mathbb{Q}} \tau$$

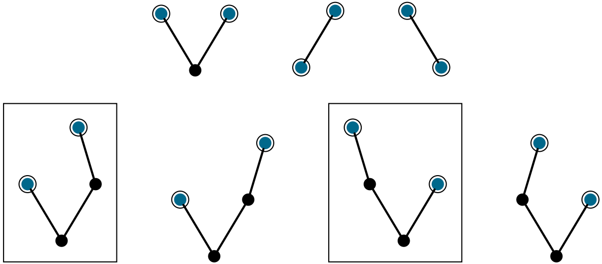
$$\equiv$$

$(\sigma \wedge \tau)0 \preceq \sigma$
or $(\sigma \wedge \tau)1 \preceq \tau$



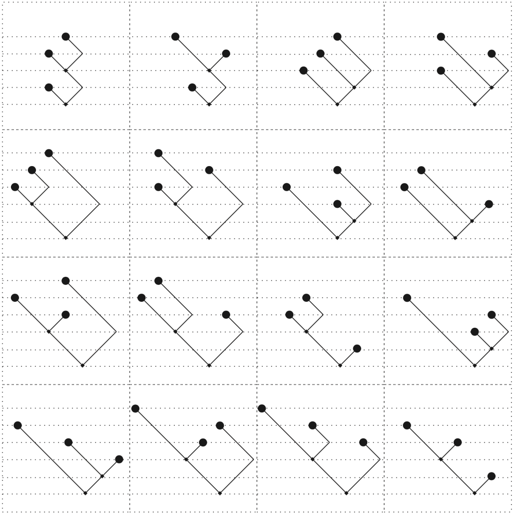
$(2^{<\omega}, <_{\mathbb{Q}})$ is a DLO

Look at the embedding types of pairs of nodes



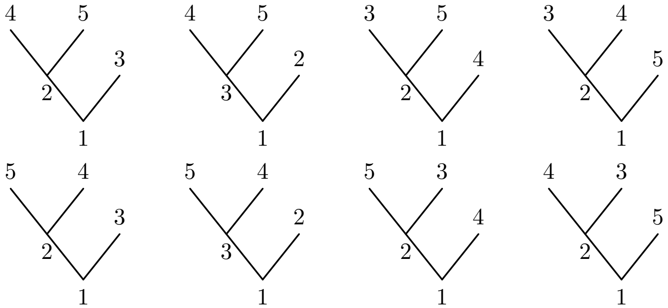
Devlin types $\equiv$ unavoidable types

Devlin types for triples



(© Introduction to Ramsey Spaces, Todorcevic)

Joyce trees with 3 leaves



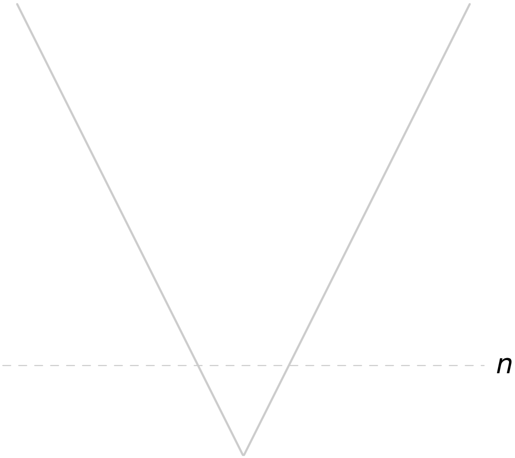
(8 more by symmetry)

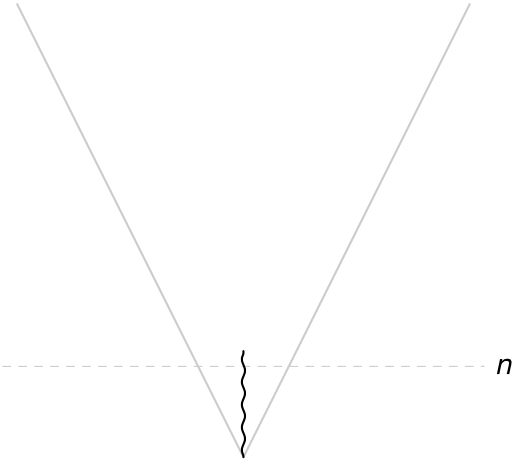
Let $\mathcal{J}_n$ be the set of Joyce trees with n leaves

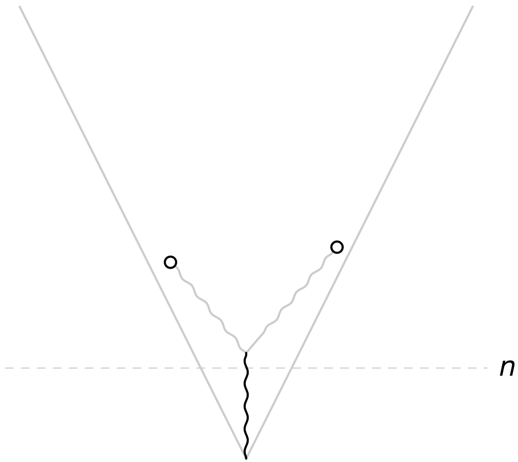
Thm (Devlin, part I)

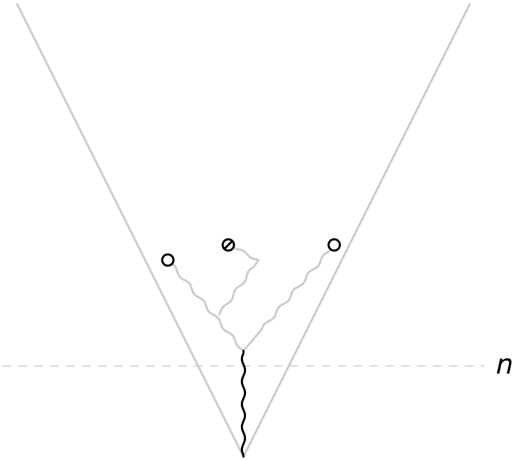
Let $f : [\mathbb{Q}]^n \to \mathcal{J}_n$ be the coloring which associates the Joyce tree. Then for every subcopy $H \subseteq \mathbb{Q}$, $[H]^n$ has all the colors.

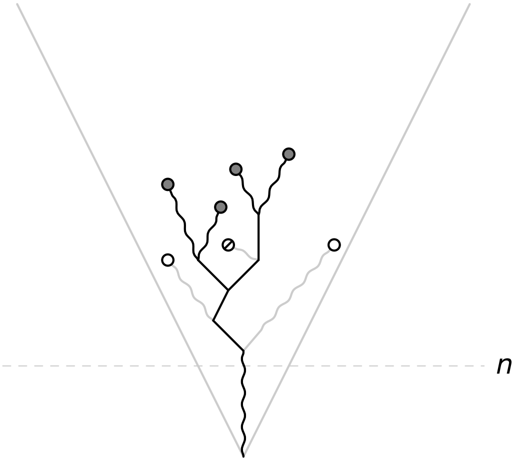
$|\mathcal{J}_0| = 1$, $|\mathcal{J}_1| = 2$, $|\mathcal{J}_3| = 16$, $|\mathcal{J}_3| = 272$


n


n


n


n


n

Devlin’s theorem

is reduced to a

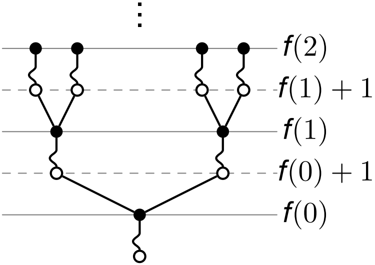
tree partition theorem

Milliken's tree theorem

Strong subtree of $2^{<\omega}$

A set $T \subseteq 2^{<\omega}$ is a tree of height $\alpha \leq \omega$ if

- every node at the same level in T has the same length;
- if $\sigma, \tau \in T$ then $\sigma \wedge \tau \in T$;
- every node which is not at level $\alpha - 1$ is 2-branching.

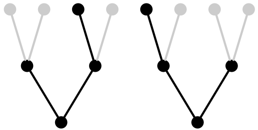


$\langle T \rangle^\alpha$: subtrees of T of height α

Thm (Milliken)

For every k-coloring of $\langle 2^{<\omega} \rangle^n$, there is a tree T of height ω such that $\langle T \rangle^n$ is monochromatic.

$\mathcal{D}_n$: Devlin types for n-tuples
$\langle T \rangle^D$: n-tuples of Devlin type D



Lem

For every $D \in \mathcal{D}_n$, there is a surjection $\iota_D : \langle 2^{<\omega} \rangle^{2n-1} \to \langle 2^{<\omega} \rangle^D$

Fix a coloring $f : [\mathbb{Q}]^n \to k$
It induces a coloring $g : [2^{<\omega}]^n \to k$

Define $h : \langle 2^{<\omega} \rangle^{2n-1} \to K$ by
$h(S) = (g(\iota_D(S)) : D \in \mathcal{D}_n)$

By Milliken's tree theorem,
$\langle T \rangle^{2n-1}$ is h-homogeneous

Embed $(\mathbb{Q}, <_{\mathbb{Q}})$ into $(T, <_{\mathbb{Q}})$
to have only Devlin types

Framework

A set S is computably P-encodable if there is a computable instance of P such that every solution computes S

Thm (Seetapun)

The computably RT^2_k-encodable sets are the computable ones

Thm (Jockusch)

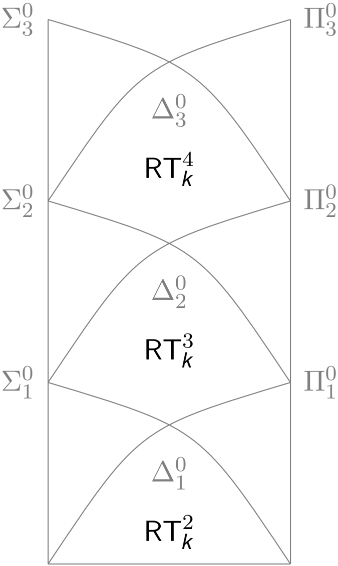
The halting set is computable RT^3_2-encodable

$$f_{\emptyset'}(x, y, z) = 1 \text{ iff } \emptyset'_y \upharpoonright x = \emptyset'_z \upharpoonright x$$

Fix some $n \geq 2$.

Thm (Cholak, Jockusch, Slaman)

The computably RT^n_k-encodable sets are the Δ^0_{n-1} ones



$$\mathsf{MTT}^n_{k,\ell}$$

Every coloring $f : \langle 2^{<\omega} \rangle^n \to k$ admits a subtree T such that $|f\langle T \rangle^n)| \leq \ell$.

$$\mathsf{DT}^n_{k,\ell}$$

Every coloring $f : [\mathbb{Q}]^n \to k$ admits a subcopy $(H, <)$ such that $|f[H]^n)| \leq \ell$.

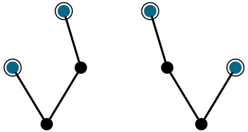
Thm (Anglès d'Auriac, Cholak, Dzhafarov, Monin, P.)

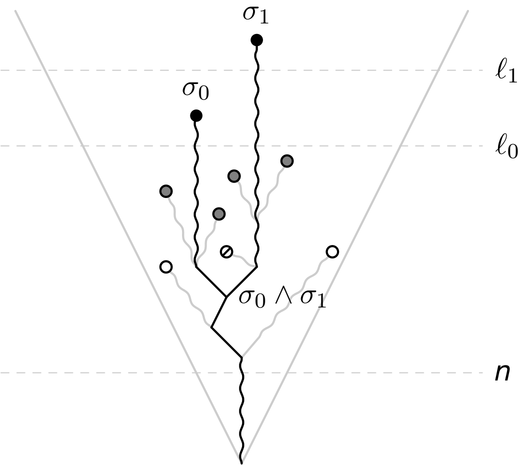
The halting set is computably $\mathrm{DT}^2_{4,3}$-encodable

$$f_{<_{\mathbb{Q}}}(\sigma,\tau) = 1 \text{ iff } |\sigma| < |\tau| \iff \sigma <_{\mathbb{Q}} \tau$$

$$f_{\emptyset'}(x,y,z) = 1 \text{ iff } \emptyset'_y \upharpoonright x = \emptyset'_z \upharpoonright x$$

$$f(\sigma,\tau) = (f_{<_{\mathbb{Q}}}(\sigma,\tau), f_{\emptyset'}(|\sigma \wedge \tau|, |\sigma|, |\tau|))$$





σ_1
ℓ_1
σ_0
ℓ_0
$\sigma_0 \wedge \sigma_1$
n

Thm (Anglès d'Auriac, Cholak, Dzhafarov, Monin, P.)

The computably $\mathsf{MTT}^3_{3,2}$-encodable sets are the computable ones

Thm (Anglès d'Auriac, Cholak, Dzhafarov, Monin, P.)

The computably $\mathsf{DT}^2_{5,4}$-encodable sets are the computable ones

Conclusion

The computational content of theorems is closely related to their combinatorics

Devlin's theorem computes through sparsity

References

Paul-Elliot Anglès d'Auriac, Peter A Cholak, Damir D Dzhafarov, Benoît Monin, and Ludovic Patey.
Milliken's tree theorem and its applications: a computability-theoretic perspective.
Memoirs of the American Mathematical Society.
to appear.

Denis Campau Devlin.
Some partition theorems and ultrafilters on omega.
ProQuest LLC, Ann Arbor, MI, 1980.
Thesis (Ph.D.)–Dartmouth College.