Exploring the abyss in Reverse Mathematics

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Thirdly, we may discuss foundational implications, though ...

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Harvey Friedman & Steve Simpson (courtesy of MFO).

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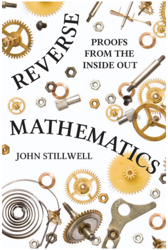

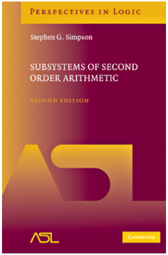

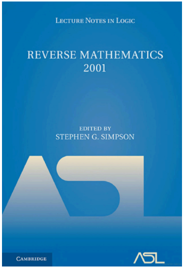

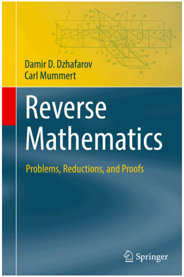

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Mummert: a few equivalences for Π^1_2-comprehension and topology.

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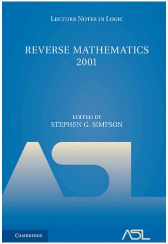

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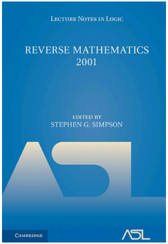

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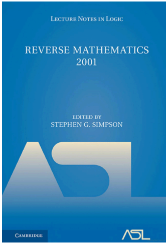

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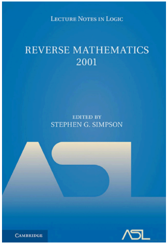

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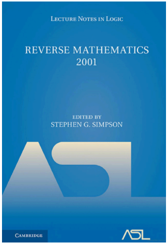

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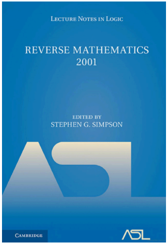

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Kohlenbach's higher-order RM uses the richer language of all finite types. Thus, the use of codes or representations is seriously reduced. E.g. discontinuous functions on $\mathbb{R}$ are directly available.

The base theory RCA_0^ω

RCA_0^ω makes use of the language of finite types: $n \in \mathbb{N}$ or n^0, $f \in \mathbb{N}^\mathbb{N}$ or f^1, $Y : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ or Y^2, et cetera.

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Real numbers and '$=_\mathbb{R}$' defined as in RCA_0; $\mathbb{R}\to\mathbb{R}$-functions are $\mathbb{N}^\mathbb{N}\to\mathbb{N}^\mathbb{N}$-functions extensional relative to '$=_\mathbb{R}$'.

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WILD: there are $2^{\mathfrak{c}}$ non-measurable quasi-continuous functions and $2^{\mathfrak{c}}$ non-Borel bounded and measurable quasi-continuous functions.

Arithmetical comprehension

The following are equivalent to ACA_0 over RCA_0:

- Let $F : C \to \mathbb{R}$ be continuous where $C \subset [0, 1]$ is an RM-closed set. Then $\sup_{x \in C} F(x)$ exists.
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Baire (1905) notes that Baire 2 functions can be represented as iterated limits.

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These third-order thms are equivalent to Π_1^1-CA_0 over $RCA_0^\omega + X$:

- For any $x \in \mathbb{N}^\mathbb{N}$, any bounded $\Sigma_1^{1,x}$-class in $\mathbb{Q}^+$ has a supremum.
- A bounded effectively Baire 2 $f : [0,1] \to \mathbb{R}$ has a supremum.
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There are a gazillion variations of the previous examples, as there are a gazillion function classes with gazillion different definitions.

The Biggest Five phenomenon of higher-order RM

Recently, Dag Normann and I have obtained a plethora of equivalences (over RCA_0^ω) between:

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- third-order theorems about (slightly) discontinuous functions.

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There are however hard limits to the Biggest Five phenomenon, with interesting consequences.

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This third-order thm is equivalent to Π^1_1-CA_0 over $\mathsf{RCA}_0^\omega + X$:

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On Kleene's arithmetical quantifier $\exists^2$

The above was obtained based on the RM of Kleene's $\exists^2$:

$$(\exists E)(\forall f \in \mathbb{N}^{\mathbb{N}})(E(f) = 0 \leftrightarrow (\exists n \in \mathbb{N})(f(n) = 0)). \qquad (\exists^2)$$

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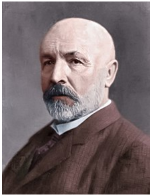
The following is not provable in $\mathsf{RCA}_0^\omega + (\exists^2) + \mathsf{Z}_2$:

There is a $\mathbb{R} \to \mathbb{R}$-function that is not Baire 2.

Exploring the abyss: the uncountability of $\mathbb{R}$

Cantor's first set theory paper (1874): uncountability of $\mathbb{R}$.



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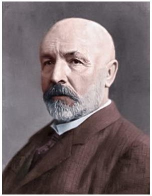

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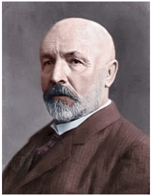

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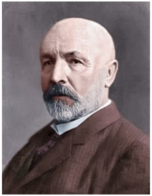

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Many many many (third-order) mainstream theorems imply NIN or NBI. However, NIN and NBI cannot be proved in $\mathsf{RCA}_0^\omega + \mathsf{Z}_2$ and stronger (higher-order) systems (see Normann-Sanders, JSL, 2022).

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Many equivalences for NIN and basic properties of regulated functions. Same for basic properties of measure and category and semi-continuity (Baire, Volterra, . . .).

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- the mercurial nature of the cardinality of $\mathbb{R}$,
- basic properties of the Lebesgue measure and integral,
- the special role of the Axiom of Choice,
- the asymmetry between measure and category.

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- the mercurial nature of the cardinality of $\mathbb{R}$:
  - ZFC cannot prove the Continuum Hypothesis.
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Thanks!

Questions?

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