Simultaneous Cupping and Continuity in Bounded Turing Degrees

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Assumed:

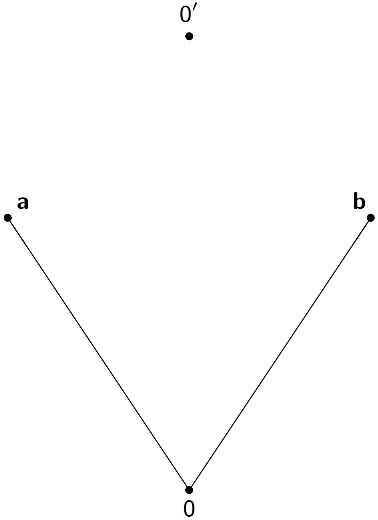
- Computable (decidable) sets, Computably enumerable (semi-decidable) sets
- Turing reduction, no constraint to query to oracles
- Bounded Turing reduction, i.e., weak-truth-table reduction, query to oracles are recursively bounded
- Turing degrees, Bounded Turing degrees
- c.e. degrees, a upper-semi-lattice
  - $\mathbf{a} \vee \mathbf{b}$ always exists, but not for $\mathbf{a} \wedge \mathbf{b}$
  - Friedberg-Muchinik theorem, Sacks's theorems: splitting and density, Lachlan's nonsplitting theorem
  - High/Low hierachy, Cups and Caps
  - Model-theoretic properties of degree structures
- c.e. bT-degrees, , also a upper-semi-lattice
  - Ladner and Sasso's splitting theorem
  - Contiguous degrees and characterizations

Shoenfield's Conjecture

As an upper-semi lattice, the structure of c.e. degrees is countably categorical.

If the conjecture is true, the theory of this structure is decidable.

Lachlan and Yates first proved the existence of minimal pairs, refuting Shoenfield's Conjecture.



Embedding of M_3 and N_5 into c.e. degrees

Lachlan's embeddings

Two nondistributive lattices, M_3 and N_5, can be embedded into c.e. degrees.

- The construction of M_3 in c.e. degrees is fairly complicated, as we need to handle with sequences of traces, which changes a lot.
- M_3 and N_5 cannot be embedded into c.e. bT-degrees, as c.e. bT-degrees form a distributive upper-semilattice.

Lachlan-Soare's example of nonembedding

The nondistributive lattice, S_8, cannot be embedded into c.e. degrees. So,

- Top elements of any M_3 in c.e. degrees are nonbranching.
- Top elements of N_5 can be branching (Ambos-Spies, Fejer).

The structure of c.e. bT-degrees admits nice algebraic properties, but not model-theoretic properties.

Contiguous Degrees

Many complicated constructions in c.e. degrees cannot be done in c.e. bT-degrees, due to the constraints on the queries.

- Lachlan's nonslitting theorem cannot be done in c.e. bT-degrees (Ladner and Sasso)

  That is, for any c.e. bT-degrees $\mathbf{a} > \mathbf{b}$, there are c.e. bT-degrees $\mathbf{a}_1, \mathbf{a}_2$ above $\mathbf{b}$, incomparable, with $\mathbf{a}_1 \vee \mathbf{a}_2 = \mathbf{a}$.
  As a consequence, a c.e. degree either contains exactly one c.e. bT-degree, or infinitely many c.e. bT-degrees.

Definition:

A c.e. degree $\mathbf{a}$ is contiguous if $\mathbf{a}$ contains exactly one c.e. bT-degree. A c.e. degree is noncontiguous if it is not contiguous.

- Array computable degrees are contiguous and contiguous degrees are low$_2$.
- Nonlow$_2$ c.e. degrees are noncontiguous.

Theorem:

Contiguous degrees, and also noncontiguous degrees, are downward dense in r.e. degrees.

A direct construction of contiguous degree

Note that $\mathbf{0}$ is contiguous.

Construct a c.e. set A, not computable, satisfying the following requirements:

P_e: $\overline{A} \neq W_e$

R_e: If $A = \Phi_e^{W_e}$ and $W_e = \Psi_e^A$, then there are *wtt*-reductions Γ_e and Δ_e such that

$$A = \Gamma_e^{W_e} \quad \text{and} \quad W_e = \Delta_e^A.$$

Here, a wtt-reduction, Γ say, is a partial computable functional, with a recursive function bounding the queries.

Techniques:

- Dumping and Confirmation
- Tree argument

Downey's Theorem: There is a contiguous nonbranching degree.

Characterizations of contiguous degrees

As c.e. bT-degrees form a distributive upper semi-lattice, contiguous degrees cannot be top elements of either M_3 or N_5 in c.e. degrees.

Theorem (Downey and Lempp (1997); Ambos-Spies and Fejer (2001)):

The following are equivalent for a c.e. degree $\mathbf{a}$:

1. $\mathbf{a}$ is contiguous;
2. $\mathbf{a}$ is locally distributive;
3. $\mathbf{a}$ is locally modular;
4. $\mathbf{a}$ cannot be the top of any N_5.

Here,

- $\mathbf{a}$ is locally distributive if for any $\mathbf{b} \leq \mathbf{a}$ and $\mathbf{a}_1$, $\mathbf{a}_2$ with $\mathbf{a}_1 \vee \mathbf{a}_2 = \mathbf{a}$, there are $\mathbf{b}_1$, $\mathbf{b}_2$ with $\mathbf{b}_1 \vee \mathbf{b}_2 = \mathbf{b}$ and $\mathbf{b}_1 \leq \mathbf{a}_1$, $\mathbf{b}_2 \leq \mathbf{a}_2$.
- $\mathbf{a}$ is locally modular if for any $\mathbf{b} \leq \mathbf{a}$ and any $\mathbf{a}_1$, $\mathbf{a}_2$ with $\mathbf{a}_1 \vee \mathbf{a}_2 = \mathbf{a}$ and $\mathbf{a}_1 \leq \mathbf{b}$, there are $\mathbf{b}_1$, $\mathbf{b}_2$ with $\mathbf{b}_1 \vee \mathbf{b}_2 = \mathbf{b}$ and $\mathbf{b}_1 \leq \mathbf{a}_1$, $\mathbf{b}_2 \leq \mathbf{a}_2$.

Locally noncappable degrees

Theorem: (Downey, Stob)

Any nonzero c.e. degree **a** bounds a nonzero c.e. degree **c** such that **c** is noncappable below **a**.

Definition: (Seetapun)

A nonzero c.e. degree **a** is locally noncappable if there is a c.e. degree **c** above **a** such that no nonzero c.e. degree below **c** can form a minimal pair with **a**.

c witnesses that **a** is locally noncappable.

Theorem: (Seetapun, 1991)

Each nonzero incomplete c.e. degree **a** is locally noncappable.

Corollaries:

- Dual of the major subdegree problem is true.
  This implies the Continuity of capping (Harrington and Soare, 1989)
- There is no maximal nonbounding degree, as when **a** is nonbounding, so is **c**.

Constructions become easier, in $\mathcal{R}_{bT}$

Here are some examples:

Theorem (Brodhead, Li and Li, 2008):

For a given nontrivial c.e. bT-degree $\mathbf{c}$, there is a c.e. bT-degree $\mathbf{a} > \mathbf{c}$ such that for any c.e. bT-degree $\mathbf{x}$, $\mathbf{c} \wedge \mathbf{x} = \mathbf{0}$ iff $\mathbf{a} \wedge \mathbf{x} = \mathbf{0}$.

The same statement is also true for c.e. degrees, proved by Seetapun in his thesis.

Theorem (Li, Li, Pan and Tang, 2008):

There are nontrivial c.e. bT-degrees $\mathbf{a}$ and $\mathbf{c}$ such that for any c.e. bT-degree $\mathbf{x}$, $\mathbf{c} \vee \mathbf{x} = \mathbf{0}'$ iff $\mathbf{x} \geq \mathbf{a}$.

That is, there are no major-subdegree in c.e. bT-degrees.

A hierarchy of cuppable degrees, with high/low-ness considered

Theorem (Ambos-Spies, Jockusch, Shore and Soare, 1984):

A c.e. degree **a** is low-cuppable (to $\mathbf{0}'$) iff **a** is noncappable.

Theorem (Li, Wu and Zhang, 2000):

There exists a cappable (and hence cannot be low-cuppable), low_2 cuppable degree **c**.

- This gives rise to a hierarchy of cuppable degrees.

$$\mathrm{L_1CUP} \subsetneq \mathrm{L_2CUP} \subseteq \mathrm{L_3CUP} \subseteq \cdots \subseteq \text{H-CUP} \subseteq \mathrm{CUP}.$$

Theorem (Greenberg, Ng and Wu, 2020):

There exists a cuppable degree **a** which is only high-cuppable.

Thus,

$$\text{only high-CUP} \subseteq \text{H-CUP} \subseteq \mathrm{CUP}.$$

$$\mathrm{L_1CUP} \subsetneq \mathrm{L_2CUP} \subseteq \mathrm{L_3CUP} \subseteq \cdots \subsetneq \text{H-CUP} \subseteq \mathrm{CUP}.$$

Question: true or fals for the following

$$\mathrm{L_2CUP} \subsetneq \mathrm{L_3CUP}; \qquad \mathrm{L_3CUP} = \bigcup_{n\geq 1} \mathrm{L}_n\mathrm{CUP}.$$

High/Low hierarchy

- Jump operator - $A' = \{e : \Phi_e^A(e) \text{ converges}\}$.

$0'$

high

high_2

low_2

low

0

Another view of cuppable degrees

Definition:

A pair of cuppable degrees, **a** and **b** is said to be simultaneously noncuppable, if there is no incomplete c.e. degree **x** such that

$$\mathbf{a} \vee \mathbf{x} = \mathbf{b} \vee \mathbf{x} = \mathbf{0}'.$$

Theorem (Li, Wu and Yang (2006)):

Such simultaneously noncuppable pairs exist. This implies that the diamond lattice can be embedded into the quotient structure $\mathcal{R}/\mathrm{NCup}$ preserving 0 and 1.

Construct c.e. sets A and B such that

- A and B are both incomplete and cuppable;
- If W cups both A and B to K, then W is Turing equivalent to K.

Recently, Tran generalizes this idea to n many degrees, $\mathbf{a}_1, \cdots, \mathbf{a}_n$, such that any $n-1$ of them are simultaneously noncuppable, but no incomplete c.e. degree can cup all of them simultaneously to $\mathbf{0}'$.

We apply the same idea to c.e. bT-degrees.

Filters and Ideals in $\mathcal{R}_{bT}$, $\mathcal{R}_{bT}/\mathrm{M}_{bT}$, $\mathcal{R}_{bT}/\mathrm{NCup}_{bT}$

Theorem (Ambos-Spies, 1985)

In $\mathcal{R}_{bT}$, $NCap_{bT}$, a strong filter, and its complement M_{bT}, an ideal, form an algebraic decomposition.

Ambos-Spies also pointed out that noncappable may not coincide with low-cuppable in the c.e. bT-degrees. So the ideal NCup_{bT} may not be a subideal of M_{bT}.

Theorem:

Simultaneously noncuppable pairs also exist in $\mathcal{R}_{bT}$. Thus, the diamond lattice can be embedded into the quotient structure $\mathcal{R}_{bT}/\mathrm{NCup}_{bT}$ preserving 0 and 1.

Some work in progress:

- $\mathcal{R}_{bT}/\mathrm{M}_{bT}$, no minimal pairs.
- $\mathcal{E}/\mathrm{M}$, $\mathcal{E}/\mathrm{NCup}$ (with Ng, Sorbi and Yang)

Thanks!

Contributors to c.e. bT-degrees

- Ambos-Spies, Fejor, Downey, Lachlan, Lempp, Nies, Shore, Stob
- Soare's book, Odifreddis' Volume II