

Primitive Recursive Algebraic Structures, and the Theory of Numberings

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Computable algebraic structures

In the 1960s, Mal'tsev and Rabin initiated the systematic development of *computable structure theory* (or the theory of constructive models).

In the talk, we work only with at most countable algebraic structures \mathcal{S} . For simplicity, we assume that all our structures have finite signatures.

A structure \mathcal{S} in the signature

$$\{P_0^{n_0}, P_1^{n_1}, \dots, P_k^{n_k}; f_0^{m_0}, f_1^{m_1}, \dots, f_\ell^{m_\ell}; c_0, c_1, \dots, c_t\}$$

is **computable** if:

- ▶ the domain of \mathcal{S} is a computable subset of ω ;
- ▶ the predicates $P_i^{\mathcal{S}}$ and the operations $f_j^{\mathcal{S}}$ are computable.

Primitive recursive structures

The classical computable structure theory studies effective and algebraic content of structures within the framework of Turing computability.

- ▶ What about sub-recursive classes of algorithms?

The following notion is introduced in the paper of Mal'tsev "Constructive algebras. I" (1961).

Note that the original definition is based on the *theory of numberings*.

Definition (Mal'tsev)

A structure \mathcal{S} in the signature

$$\{P_0, P_1, \dots, P_k; f_0, f_1, \dots, f_\ell; c_0, c_1, \dots, c_t\}$$

is **primitive recursive** if:

- ▶ the domain of \mathcal{S} is a primitive recursive subset of ω , and
- ▶ the predicates $P_i^{\mathcal{S}}$ and the operations $f_j^{\mathcal{S}}$ are primitive recursive.

Eliminating unbounded search

The *restricted Church–Turing Thesis* for primitive recursive functions states the following:

A function $f(\vec{x})$ is primitive recursive if and only if it can be described by an algorithm which uses only bounded loops.

Informally speaking, in a Pascal-like programming language, a program is not allowed to use instances of `WHILE ... DO`, `REPEAT ... UNTIL`, and `GO TO`.

Therefore, one can say that in primitive recursive structures, we eliminate unbounded search procedures.

We note that there are other ways to implement sub-recursive, countably infinite algebraic structures:

(i) The framework of *polynomial-time computable* structures:

- ▶ The works of Cenzer, Grigorieff, Nerode, Remmel, and other authors (1990s). For example:
 - ▶ Every computable linear order has a polynomial-time copy [Grigorieff 1990].
 - ▶ Every computable torsion abelian group has a polynomial-time copy [Cenzer and Remmel 1991].
- ▶ The recent works of Alaev and Selivanov (since 2016).

(ii) The framework of automatic structures (Khoussainov and Nerode 1995, Blumensath and Grädel 2000, ...).

(iii) Structures in the Grzegorzczuk hierarchy (Alaev and Selivanov 2021).

Punctual structures: Motivation

In the algorithm design for finite structures, there are online and “offline” algorithms:

- ▶ *Online situation*: Input arrives bit-by-bit, and decisions have to be made on the fly.
- ▶ *Offline situation*: One can make decisions after seeing all of the input data.

Example. Every finite tree T is 2-colorable, but the desired coloring is achievable only in offline fashion.

In this setting, online situation looks like this:

At a stage s , our input shows a new node v_s of the tree T . We have to declare the color of v_s *right now*, i.e. we cannot wait for v_{s+1} to appear.

It is known that in the online situation, the sharp lower bound is $O(\log n)$ colors for a tree with n vertices.

Punctual structures: Motivation

Punctual structures aim to emulate online situation for primitive recursive algorithms.

Example. Cenzer and Remmel (1991) showed that every computable relational structure \mathcal{S} has a polynomial-time isomorphic copy \mathcal{P} .

Proof sketch for obtaining a primitive recursive copy \mathcal{P} .

W.l.o.g., one may assume that $\text{dom}(\mathcal{S}) = \omega$.

Suppose that we have already chosen the first $n + 1$ elements of \mathcal{P} — a_0, a_1, \dots, a_n from ω .

Wait until all signature \mathcal{S} -relations for all tuples \bar{b} from $\{0, 1, \dots, n + 1\}$ are computed. While waiting, declare:

$$a_n + 1 \notin \text{dom}(\mathcal{P}), a_n + 2 \notin \text{dom}(\mathcal{P}), a_n + 3 \notin \text{dom}(\mathcal{P}), \dots$$

When the \mathcal{S} -relations are eventually computed, we choose a_{n+1} as the least element, which has not declared anything about its membership in $\text{dom}(\mathcal{P})$. □

The example above is not online: We just keep postponing our decision on how to deal with (a copy of) the next element from \mathcal{S} .

Punctual structures

Definition (Kalimullin, Melnikov, and Ng 2017)

A countably infinite algebraic structure \mathcal{S} is **punctual** if $\text{dom}(\mathcal{S}) = \omega$, and the signature functions and predicates of \mathcal{S} are primitive recursive.

Since 2017, the theory of punctual structures has become a vast research area with a lot of interesting results and applications.

For example, the methods of this theory are used in the proof of the following:

Theorem (B., Harrison-Trainor, Kalimullin, Melnikov, and Ng 2019)

The index set of computable structures which have a polynomial-time isomorphic copy is m -complete Σ_1^1 .

- (i) Effective categoricity for punctual structures:
The case of finitely generated structures

Computable categoricity

Definition (Mal'tsev)

A computable structure \mathcal{S} is **computably categorical** (or *autostable*) if for any computable structure \mathcal{A} isomorphic to \mathcal{S} , there is a computable isomorphism f from \mathcal{A} onto \mathcal{S} (i.e. f is an isomorphism, which is also a computable function).

Informally speaking, all computable isomorphic copies of a computably categorical structure \mathcal{S} have the same algorithmic properties (from the point of view of classical computable structure theory).

Proposition (Mal'tsev 1961)

Every finitely generated computable structure is computably categorical.

Punctual categoricity

Definition (Kalimullin, Melnikov, and Ng)

A punctual structure \mathcal{S} is **punctually categorical** if for any punctual structure \mathcal{A} isomorphic to \mathcal{S} , there is an isomorphism $f: \mathcal{A} \cong \mathcal{S}$ such that both f and f^{-1} are primitive recursive.

[Recall that Kuznecov (1950) proved that there is a primitive recursive permutation f such that its inverse f^{-1} is not primitive recursive.]

Theorem (Kalimullin and Melnikov)

If a punctual structure \mathcal{S} is punctually categorical, then \mathcal{S} is either finitely generated, or locally finite.

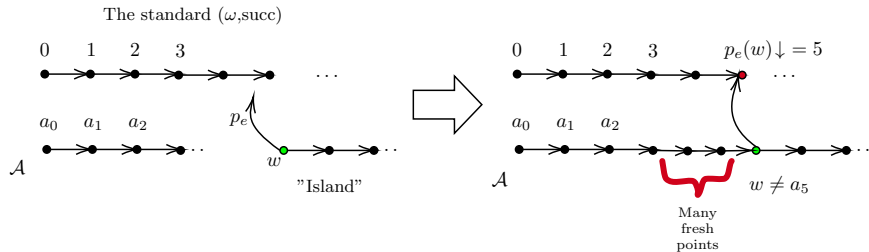
Within the punctual framework, the properties of effective categoricity for finitely generated structures become highly non-trivial.

Example. The structure $\mathcal{S} = (\omega; \text{succ})$, where $\text{succ}(x)$ is the successor function, is not punctually categorical.

Proof Sketch. Fix a computable list of all unary primitive recursive functions $(p_e)_{e \in \omega}$.

We build a punctual \mathcal{A} isomorphic to $(\omega; \text{succ})$, satisfying the following requirements:

R_e : p_e is not an isomorphism from \mathcal{A} onto (the standard copy of) \mathcal{S} .



PR-degrees

Primitive recursive isomorphisms give rise to a natural structure of degrees:

Definition (Kalimullin, Melnikov, and Ng)

Let \mathcal{A} and \mathcal{B} be isomorphic punctual structures. We say that $\mathcal{A} \leq_{pr} \mathcal{B}$ if there is a primitive recursive isomorphism f from \mathcal{A} onto \mathcal{B} .

For a punctual structure \mathcal{S} , by $\mathbf{PR}(\mathcal{S})$ we denote the poset

$$(\{\mathcal{A} : \mathcal{A} \text{ is a punctual isomorphic copy of } \mathcal{S}\} / \equiv_{pr}; \leq_{pr}).$$

We discuss results on the posets $\mathbf{PR}(\mathcal{S})$ for finitely generated (or f.g., for short) structures \mathcal{S} .

The least degree

Proposition (Kalimullin, Melnikov, and Ng)

If \mathcal{S} is a f.g. punctual structure, then $\mathbf{PR}(\mathcal{S})$ has a least element.

Proof Idea. The least degree is induced by a “naturally generated” term algebra built around the finite set of generators of \mathcal{S} . □

Density

Theorem 1 (B., Kalimullin, Melnikov, and Ng 2020)

Let \mathcal{S} be a f.g. punctual structure. Then the poset $\mathbf{PR}(\mathcal{S})$ is dense, i.e. if \mathcal{A} and \mathcal{B} are punctual copies of \mathcal{S} such that $\mathcal{A} <_{pr} \mathcal{B}$, then there is a punctual \mathcal{C} with $\mathcal{A} <_{pr} \mathcal{C} <_{pr} \mathcal{B}$.

Consequently, if $\text{card}(\mathbf{PR}(\mathcal{S})) > 1$, then $\mathbf{PR}(\mathcal{S})$ is countably infinite.

[Cf. a similar result on punctual numberings (to be discussed below).]

Theorem (Greenberg, Harrison-Trainor, Melnikov, and Turetsky 2021)

There exists a punctual structure \mathcal{M} such that $\mathbf{PR}(\mathcal{M})$ is not dense.

Greatest element

Proposition (Kalimullin, Melnikov, and Ng)

The poset $\mathbf{PR}(\omega; \text{succ})$ has no maximal elements.

Theorem 2 (B., Kalimullin, Melnikov, and Ng 2020)

There exists a punctual f.g. structure \mathcal{S} such that the poset $\mathbf{PR}(\mathcal{S})$ is infinite and has a greatest element.

Embeddings of countable lattices

Theorem (Kalimullin, Melnikov, and Zubkov)

Let \mathcal{S} be a *rigid*, f.g. punctual structure such that $\text{card}(\mathbf{PR}(\mathcal{S})) > 1$.
Then for a countable lattice L , the following are equivalent:

- ▶ L is isomorphically embeddable into $\mathbf{PR}(\mathcal{S})$ (preserving suprema and infima),
- ▶ L is distributive.

(ii) Rogers semilattices for punctual numberings

Numberings: Reducibilities

Let S be a countable set. A **numbering** ν of the set S is a surjective map from the set of natural numbers ω onto S .

A numbering ν is **reducible** to a numbering μ , denoted by $\nu \leq \mu$, if there is a total computable function $f(x)$ such that

$$\nu(n) = \mu(f(n)) \text{ for all } n \in \omega.$$

Numberings ν and μ are **equivalent** (denoted by $\nu \equiv \mu$) if $\nu \leq \mu$ and $\mu \leq \nu$.

Definition

We say that a primitive recursive function f punctually reduces ν to μ (denoted by $f: \nu \leq_{pr} \mu$), if

$$\nu(n) = \mu(f(n)), \text{ for all } n.$$

The preorder \leq_{pr} induces the corresponding equivalence relation \equiv_{pr} on numberings.

Numberings: Computable and punctual

Let ν be a numbering of a family $S \subset P(\omega)$. The numbering ν is **computable** if the set

$$G_\nu = \{(n, x) : x \in \nu(n)\}$$

is computably enumerable. In other words, ν uniformly enumerates the family S consisting of c.e. sets.

A family $S \subset P(\omega)$ is *computable* if S has a computable numbering.

By $Com_{\Sigma_1^0}(S)$ we denote the set of all computable numberings of the family S .

Definition

Let S be a family of primitive recursive functions. A numbering ν of the family S is punctual if the function

$$g_\nu(n, x) := (\nu(n))(x)$$

is primitive recursive.

A family S is *punctual* if it has a punctual numbering.

By $Com_{pr}(S)$ we denote the set of all punctual numberings of S .

Numberings: Rogers semilattices

Given numberings ν and μ of a family S , one defines a new numbering $\nu \oplus \mu$ as follows.

$$(\nu \oplus \mu)(2n) := \nu(n), \quad (\nu \oplus \mu)(2n + 1) := \mu(n).$$

- ▶ *In the classical setting:* For a computable family S , the quotient structure

$$\mathcal{R}_{\Sigma_1^0}(S) := (Com_{\Sigma_1^0}(S); \leq, \oplus) / \equiv$$

is an upper semilattice. It is called the **Rogers semilattice** of the computable family S .

- ▶ *In the punctual setting:* For a punctual family S , the structure

$$\mathcal{R}_{pr}(S) := (Com_{pr}(S); \leq_{pr}, \oplus) / \equiv_{pr}$$

is the (punctual) **Rogers semilattice** of the punctual family S .

Minimal elements

In the classical setting:

- ▶ For the family of sets $\mathcal{S}_0 = \{\emptyset, \{0\}\}$, the Rogers semilattice $\mathcal{R}_{\Sigma_1^0}(\mathcal{S}_0)$ is isomorphic to the semilattice \mathbf{R}_m of c.e. m -degrees. Notice that this semilattice has a least element.
- ▶ Let \mathcal{CE} be the family of all c.e. sets. The semilattice $\mathcal{R}_{\Sigma_1^0}(\mathcal{CE})$ has infinitely many minimal elements:
 - ▶ induced by Friedberg (i.e. 1–1) numberings [Rogers 1967];
 - ▶ induced by positive undecidable numberings [Ershov 1968];
 - ▶ induced by non-positive numberings [Khutoretskii 1969].

In the punctual setting:

Theorem 3 (B., Mustafa, and Ospichev 2020)

Let \mathcal{S} be a punctual family.

- ▶ If \mathcal{S} is finite, then the punctual Rogers semilattice $\mathcal{R}_{pr}(\mathcal{S})$ is one-element.
- ▶ If \mathcal{S} is infinite, then $\mathcal{R}_{pr}(\mathcal{S})$ has no minimal elements.

Density

In the classical setting:

For the family $\mathcal{S}_0 = \{\emptyset, \{0\}\}$, the semilattice $\mathcal{R}_{\Sigma_1^0}(\mathcal{S}_0)$ is isomorphic to \mathbf{R}_m .

Hence, the semilattice has atoms (\Leftarrow minimal c.e. m -degrees).

In the punctual setting:

Theorem 4 (B., Mustafa, and Ospichev 2020)

Let \mathcal{S} be an infinite punctual family. Then the punctual Rogers semilattice $\mathcal{R}_{pr}(\mathcal{S})$ is dense, i.e. for any punctual numberings $\nu <_{pr} \mu$ of the family \mathcal{S} , there is a numbering $\xi \in Com_{pr}(\mathcal{S})$ such that

$$\nu <_{pr} \xi <_{pr} \mu.$$

Informally speaking, this theorem is *the same* as the result on density of $\mathbf{PR}(\mathcal{A})$ for a f.g. structure \mathcal{A} , but with all “algebraic” details omitted.

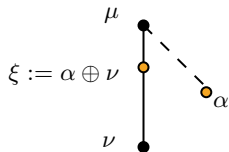
Proof sketch for Theorem 4 (if time permits)

For any punctual numberings $\nu <_{pr} \mu$ of our family \mathcal{S} , there is a numbering ξ such that $\nu <_{pr} \xi <_{pr} \mu$.

Proof Sketch. Fix a computable list of all unary p.r. functions $(p_e)_{e \in \omega}$.

We build a punctual numbering α of some subfamily $\mathcal{S}_0 \subseteq \mathcal{S}$.

- ▶ At a stage s , we promptly define a number $g(s)$, and declare $\alpha(s) = \mu(g(s))$. This will ensure that $\alpha \leq_{pr} \mu$.
- ▶ The desired ξ is defined as $\alpha \oplus \nu$. This guarantees that $\nu \leq_{pr} \xi \leq_{pr} \mu$.



Requirements:

$R_e : p_e$ does not reduce α to ν .

$Q_i : p_i$ does not reduce μ to $\alpha \oplus \nu$.

$R_e : p_e$ does not reduce α to ν .

$Q_i : p_i$ does not reduce μ to $\alpha \oplus \nu$.

R_e-strategy. Suppose that the strategy starts working at a stage s_0 . Our numbering α “copies” the top numbering μ :

$$\alpha(s_0) := \mu(0), \quad \alpha(s_0 + 1) := \mu(1), \quad \alpha(s_0 + 2) := \mu(2), \quad \dots,$$

until we find a witness $v \in \omega$ such that $\alpha(v) \neq \nu(p_e(v))$.

Such a witness will be eventually found, since:

- (a) The function $p_e(x)$ is total.
- (b) If v is never found, then one can deduce that $\mu \leq_{pr} \nu$, which contradicts the condition $\nu <_{pr} \mu$.

When v is found, the requirement R_e is forever satisfied.

Q_i-strategy. Suppose that it starts working at a stage t_0 . The numbering α “copies” the bottom numbering ν :

$$\alpha(t_0) := \nu(0), \quad \alpha(t_0 + 1) := \nu(1), \quad \alpha(t_0 + 2) := \nu(2), \quad \dots,$$

until we find a witness v such that $\mu(v) \neq (\alpha \oplus \nu)(p_i(v))$.

If such a witness is never found, then one would be able to show that $\mu \leq_{pr} \nu$. □

Lattices

In the classical setting:

- ▶ Let \mathcal{S} be a computable family. Then the Rogers semilattice $\mathcal{R}_{\Sigma_1^0}(\mathcal{S})$ is either one-element, or countably infinite. [Khutoretskii 1971]
- ▶ If the semilattice $\mathcal{R}_{\Sigma_1^0}(\mathcal{S})$ is infinite, then it is not a lattice. [Selivanov 1976]

In the punctual setting:

Proposition (B., Mustafa, and Ospichev)

Let \mathcal{S} be an infinite punctual family satisfying the following condition: If ν is an arbitrary punctual numbering of \mathcal{S} , then its equivalence relation η_ν , where

$$(k \eta_\nu \ell) \Leftrightarrow \nu(k) = \nu(\ell),$$

is primitive recursive. Then the structure $\mathcal{R}_{pr}(\mathcal{S})$ is an infinite lattice.

Example. The family of constant functions

$$\mathcal{S} = \{\lambda x.i \mid i \in \omega\}$$

satisfies the conditions of the proposition.

The framework of punctual numberings is connected with:

- ▶ The theory of punctual structures: Some proofs for structures can be transferred into the setting of numberings.
- ▶ Primitive recursive (punctual) equivalences, under primitive recursive reducibility [B., San Mauro, Sorbi, and Ng].

Roughly speaking, these are similar to the classical connections:

- ▶ Theory of computable numberings \leftrightarrow computable structure theory:
 - ▶ For example, the classical results of Goncharov (1980):
 - For any finite $N \geq 2$, there is a computable family \mathcal{S} which has precisely N pairwise non-equivalent, computable Friedberg numberings.
 - For any $N \geq 2$, there is a computable structure with computable dimension N .
- ▶ Theory of numberings \leftrightarrow positive equivalences, and computable reducibility on them.

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