Computable groups and computable group orderings

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Linear orderings on groups

Definition of *bi-orderable* groups

Let G be a group and $<$ be a linear order on G. G is said to be *bi-orderable* with respect to $<$ if for each $g, h, x \in G$ if $g \leq h$, then

1. $xg \leq xh$,
2. $gx \leq hx$.

In the above definition if only Condition 1 necessarily holds, then G is said to be *left-orderable* with respect to $<$.

A naturally associated concept with group orders is *positive cone* that can be defined as follows:

$$PC(G, <) := \{g \in G \mid g > 1\}$$

Some remarks

- The bi-orderings on groups gained popularity after seminal works of Dedekind, Hölder, and Hilbert, where they were considering bi-orderings in a broad algebraic context;
- In more abstract group theoretical context bi-orderable groups where intensively studied starting from 1940's by Levi, B.Neumann, and others.
- Left-orderable groups have more modern origin. However, due to their natural occurrence in groups' classes with interesting geometric, topological, and dynamical properties, in recent years, they gained broad popularity. For example, we have

Theorem

A countable group G is left-orderable if and only if it embeds into $Homeo^+(\mathbb{R})$, the group of orientation preserving homeomorphisms of $\mathbb{R}$.

Computable groups-1

Interactions between *combinatorial group theory* and *computability theory* has a long history that goes back to the seminal work of Max Dehn from 1911, where he introduced word, conjugacy, and isomorphism problems in *finitely generated* groups. The highest points in this area are the theorems of Higman and Boone-Higman that correspondingly state:

- (Higman, 1961) A given finitely generated group has a recursive presentation if and only if it *embeds* into a finitely presented group;
- (Boone-Higman, 1974) A finitely generated group has decidable word problem if and only if it embeds into a simple subgroup of a finitely presented group.

Computable groups-2

Seminal works of Fröhlich-Shepherdson, Rabin, and Mal'cev, done in 1950's and 1960's, significantly extended the scope of algebraic structures the computability properties of which were of interest. In particular, the analog of groups with decidable word problem for countable (but not necessarily f.g.) groups was introduced (independently) by Rabin and by Mal'cev.

Definition (Rabin, 1960; Mal'cev, 1961)

A presentation $G = \langle X \mid R \rangle$ of a countable group is called *computably enumerated* if the sets X and $R \subseteq (X \cup X^{-1})^*$ are computably enumerated. It is said that $G = \langle X \rangle$ is a *computable group* with respect to the computably enumerated generating set X if the set

$$\{u \in (X \cup X^{-1})^* \mid u =_G 1\}$$

is computable.

Computable orders on groups

In the context of computability theory on algebraic structures, it is very natural to consider computability properties of structures associated with orderings on groups. In particular, in 1986, Downey and Kurtz initiated a systematic study of computability theory of positive cones of ordered groups.

Definition (Computable orders)

Let G be a (countable) group and $<$ be a linear order on it. Then, $<$ is said to be computable with respect to the given presentation $G = \langle X \mid R \rangle$ if

- G is computable with respect to that presentation, and
- $PS(G, <)$ is computably enumerable.

In other words, X is computably enumerated and for any $w \in (X \cup X^{-1})^*$ one can algorithmically realize whether $w >_G 1$, $w =_G 1$, or $w <_G 1$.

Extensive work on studies of computability properties of group orders has been done by many authors. A recent advancement includes the following:

Theorem (Antolin, Rivas, Su, 2021)

A finitely generated left-orderable group has only regular left orders if and only if it is **poly-$\mathbb{Z}$ Tararin group**.

Remark: A group G is *poly-$\mathbb{Z}$ Tararin* if and only if there exists a unique subnormal series $G = G_0 \rhd G_1 \rhd \ldots \rhd G_n = 1$ such that for all i, $G_i/G_{i+1} \simeq \mathbb{Z}$ and $G_i/G_{i+2} \simeq K$, where $K = \langle a, b \mid aba^{-1} = b^{-1} \rangle$ is the Klein bottle (fundamental) group.

In particular, if all left-order of a f.g. l.-o. group are regular, then the group must be **solvable**.

Downey-Kurtz's question

G is said to be *computably (bi- or left-) orderable* if it possesses a (bi- or left-) order $<$ and a presentation with respect to which $<$ is computable.

Question of Downey and Kurtz, 1999

Is every computable orderable group isomorphic to computably orderable group?

For abelian groups, a positive answer to the question of Downey and Kurtz was obtained by Reed Solomon in 2002.

Theorem (R. Solomon, 2002)

Every bi-orderable computable abelian group possesses a presentation with computable bi-order.

In case of left-orderable groups, Harrison-Trainor showed that, in general, the answer to the question is negative.

Theorem (Harrison-Trainor, 2018)

There exists a computable left-orderable group G that does not possess a computable left-order with respect to any presentation of G.

Harrison-Trainor's result extends to the general case of bi-orderable groups in a stronger form.

Theorem (D., 2020)

There exists a two-generated bi-orderable computable group G that does not embed in any countable group with a computable left- or bi- order. Moreover, G can be chosen to be a **solvable group** *of derived length 3.*

Question. Does there exist a computable bi-orderable metabelian group that does not possess a computable bi-order?

Finitization theorems - 1

Theorem (A characterization of computable groups, D., '20)

A countable group H is computable if and only if it is a subgroup of a finitely-generated group with decidable word problem such that the membership problem for the subgroup H is decidable in the large group.

Theorem (A characterization of computable orders, D., '20)

A countable group H has a computable left- or bi- ordering if and only if it is a subgroup of a finitely-generated group with computable left- or bi- order, respectively, such that the membership problem for the subgroup H is decidable in the large group.

Moreover, for any fixed computable order on H we can assume that the large group continues the order on H.

Finitization theorems - 2

Theorem (D.-Steenbock, 2020)

A **countable group** *with computable left-order is a subgroup of a* **finitely generated simple** *left-orderable group with computable left-order.*

Remarks: Until recently the existence of finitely generated left-orderable simple groups was unknown and was regarded as a well-known open question in group theory. The first examples of such groups were discovered by Hyde and Lodha in 2019.

Finitization theorems - 3

A landmark "finitization theorem" in (combinatorial) group theory is the theorem of Boone-Higman and R.Thompson.

Theorem (Boone-Higman, 1974; R.Thompson, 1980)

A finitely generated group has decidable word problem if and only if it is a subgroup of a **finitely generated simple** *subgroup of a* **finitely presented** *group.*

Bludov and Glass showed that a version of this theorem for left-order groups holds, except that they didn't show that the simple group can be taken to be finitely generated.

For **countable** groups with **recursively enumerable** left-order, one has the following:

Theorem (Bludov-Glass, 2009; D.-Steenbock, 2020)

A countable group with recursively enumerable left-order has computable left-order if and only if it is a subgroup of a **finitely generated simple** *subgroup of a finitely presented left-orderable group.*

Thank you!