Amalgamation Constructions and Recursive Model Theory

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Strongly Minimal Theories

Definition

A first order theory T is strongly minimal if for every $\bar{a} \in M \models T$ and every formula $\phi(x, \bar{y})$, $\phi(x, \bar{a})$ defines a finite or co-finite subset of M.

Example

- A regular acyclic graph with finite valence (say, the Cayley graph of a finitely generated group);
- A vector space (say, $(\mathbb{Q}, +)$);
- An algebraically closed field, (say $(\mathbb{C}, +, \cdot)$)

Dimension characterizes models

In each of these examples, there is a notion of closure and dimension which characterizes models. This is not a coincidence.

Theorem (Baldwin-Lachlan)

If T is $\aleph_1$-categorical, then each model of T is determined by a single cardinal invariant, its dimension. If M is countable, then $\dim(M) \in \omega + 1$.

Zilber conjectured that in fact our canonical examples of strongly minimal theories formed an exhaustive list.

The Zilber Trichotomy

Zilber conjectured that every strongly minimal theory was of one of three types:

- Disintegrated (Essentially binary)
- Locally Modular (Essentially a vector space)
- Field-like (Essentially an algebraically closed field)

Theorem (Hrushovski 1991)

The Zilber trichotomy is false. There are non-trichotomous theories, and there are Hrushovski constructions!

The basic Hrushovski amalgamation construction 1/3

Let L be the language generated by a single ternary relation symbol R. Throughout, we will enforce that R is symmetric and anti-reflexive ($R(a, a, b)$ never holds).

For a finite L-structure A, define $\delta(A) = |A| - \#R(A)$. For a pair of finite L-structures $A \subseteq B$, $\delta(B/A) = \delta(B) - \delta(A)$. Idea: δ is an approximation to the dimension that A will have in our constructed model. Roughly speaking, we want to make B algebraic over A if $\delta(B/A) \leq 0$. To do this, we construct the following class of finite L-structures:

Definition

Let $\mathcal{C}$ be the class of finite L-structures C such that

- If $A \subseteq C$ then $\delta(A) \geq 0$.
- If $B_1, \ldots B_n$ all contain A such that $(B_i, A) \cong (B_j, A)$, $\delta(B_1/A) = 0$, and B_1 contains no subset E such that $A \subsetneq E \subsetneq B_1$ and $\delta(E/A) \leq 0$, then $n \leq 2^{|A|}$.

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This $\mathcal{C}$ forms an amalgamation class (sort of). We say $A \leq B$ if $A \subseteq B$ and $\delta(E/A) \geq 0$ whenever $A \subseteq E \subseteq B$.

Lemma

If $A, B, C \in \mathcal{C}$ such that $A \leq B$ and $A \leq C$, then there exists a $D \in \mathcal{C}$ with $B \leq D$ and $C \leq D$.

The basic Hrushovski amalgamation construction 3/3

Lemma

If $A, B, C \in \mathcal{C}$ such that $A \leq B$ and $A \leq C$, then there exists a $D \in \mathcal{C}$ with $B \leq D$ and $C \leq D$.

By repeatedly amalgamating within the class $\mathcal{C}$, we get a countable structure $\mathcal{M}$ such that

1. If $A \subset \mathcal{M}$ then $A \in \mathcal{C}$
2. If $A \leq \mathcal{M}$ and $A \leq B$, then there is an embedding $f : B \to \mathcal{M}$ over A such that $f(B) \leq \mathcal{M}$

Theorem

This $\mathcal{M}$ is unique up to isomorphism, is saturated, strongly minimal, and refutes the Zilber conjecture.

The proof is combinatorics heavy, which highlights the nature of $\mathrm{Th}(\mathcal{M})$ as combinatorial and not algebraic.

Definition

- All languages L are countable and recursive.
- An L-structure A is recursive if $|A| = \omega$ and the atomic diagram of A is recursive.
- An L-structure A is decidable if $|A| = \omega$ and the elementary diagram of A is recursive.
- A is recursively (decidably) presentable if A is isomorphic to a recursive (decidable) model.

- If T is recursive, then it has at least one decidable model (Henkin's construction).
- If A is recursive, then $T \leq_T 0^\omega$ (true arithmetic), but need not be simpler. For example, consider the theory $\text{Th}(\mathbb{N}, +, \cdot)$.

Question

Is there a tighter connection between the complexity of a theory and its models if the theory is model theoretically tame?

For example, if T is recursive and tame, must more than one model of T be decidable? Conversely, if A is recursive and model theoretically tame, then is there any better bound on the complexity of $\text{Th}(A)$? Would $\text{Th}(A)$ have to be arithmetical?

One direction works

The relationship between the complexity of a theory and its models is strong in one direction for model-theoretically nice theories.

Theorem (Harrington 1974, Khisamiev 1974)

If T is $\aleph_1$-categorical and recursive, then every countable model of T is decidably presentable.

Theorem (A. - A more general version of Harrington-Khisamiev)

Let T be ω-stable. Then all countable models of T are decidably presentable if and only if all n-types consistent with T are recursive and T has only countably many countable models up to isomorphism.

Theorem (Obvious from Henkin's construction)

If T is $\aleph_0$-categorical and recursive, then every countable model of T is decidably presentable.

The other direction begins to fail

Theorem (Goncharov-Khoussainov, 2004)

For each n, there exists an $\aleph_1$-categorical theory T so that $T \equiv_T 0^n$ and every countable model of T is recursively presentable. Similarly with $\aleph_1$-categorical replaced by $\aleph_0$-categorical.

Theorem (Fokina, 2006)

Fix $\mathbf{d}$ any arithmetical turing degree. There are $\aleph_1$-categorical theories and $\aleph_0$-categorical theories of degree $\mathbf{d}$ whose countable models are recursively presentable.

Theorem (Khoussainov-Montalban, 2010)

There exists a recursive $\aleph_0$-categorical structure A such that $\mathrm{Th}(A) \equiv_T 0^\omega$.

The complete answer to the failing direction

Observation

If T has a recursive model, then $T \leq_{tt} 0^{\omega}$.

Theorem (A.)

Let $\mathbf{d}$ be any tt-degree $\leq 0^{\omega}$. Then there exists both strongly minimal and $\aleph_0$-categorical theories with finite signatures in $\mathbf{d}$ all of whose countable models are recursively presentable.

Spectra of Strongly Minimal Theories

Recall: Baldwin-Lachlan gives us that the countable models of a strongly minimal (non-$\aleph_0$-categorical) countable theory form an $\omega + 1$-chain $M_0 \preceq M_1 \preceq \ldots \preceq M_\omega$.

Definition

Let $SRM(T) = \{n | M_n \text{ is recursively presentable}\}$.

Question

1. Which sets are spectra?
2. Which sets are spectra in finite languages?
3. Which sets are spectra of trichotomous theories? (i.e., which sets are spectra *requiring* a Hrushovski construction to achieve?)

Known Examples of Spectra

Answer

The following sets are known to be spectra:

- $\emptyset$
- $\omega + 1$
- $\{0\}$ (Goncharov 1978)
- $\{0, \ldots n\}$ (Kudaibergenov 1980)
- ω (Khoussainov, Nies, Shore 1997)
- $\omega + 1 \setminus \{0\}$ (Khoussainov, Nies, Shore 1997)
- $\{1\}$ (Nies 1999)
- $[1, \alpha)$ (Nies, Hirschfeldt unpublished)
- $\{\omega\}$ (Hirschfeldt, Khoussainov, Semukhin, 2006)
- $\{0, \omega\}$ (A.)

Known Examples of Spectra in Finite Languages

Answer

The following sets are known to be spectra in finite languages:

- $\emptyset$
- $\omega + 1$
- $\{0\}$ (Herwig, Lempp, Ziegler 1997)
- $\{0, \ldots n\}$ (A.)
- ω (A.)
- $\{\omega\}$ (A.)
- $\{0, \omega\}$ (A.)

For these results, I needed a Hrushovski construction, while each result on the last slide (aside from $\{0, \omega\}$) and $\{0\}$ here was constructed in a disintegrated theory.

Something new was necessary!

Conjecture

If T is a strongly minimal trichotomous theory in a finite language, then $SRM(T) = \emptyset, \omega + 1,$ or $\{0\}$.

Some evidence for the conjecture comes from the following:

Theorem (A.-Medvedev)

If T is a disintegrated strongly minimal theory in a finite language, then $SRM(T) = \emptyset, \omega + 1,$ or $\{0\}$.

Theorem (A.-Medvedev)

If T is a locally modular theory in a finite language which expands a group, then $SRM(T) = \emptyset, \omega + 1,$ or $\{0\}$.

Theorem (Poizat, 1988)

If T is a field-like theory in a finite language which expands a field, then $SRM(T) = \omega + 1$.

Thank you

ever so much for your patience!