On the number of infinite sequences with trivial initial segment complexity

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Question by Downey, Miller, Nies, Yu

The map G takes c to the number of K-trivial streams with constant c.

What is the arithmetical complexity of G?

. . . or equivalently

How hard is to compute G?

Trivial sequences

A stream is random if it has high initial segment complexity.

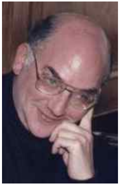
> *To describe the first n bits of the sequence you need to use n bits (modulo a constant)*

On the other end of the spectrum:

> *A stream is trivial if the complexity of its first n bits is as low as the complexity of* 0^n*.*

Chaitin and Solovay in 1975

Chaitin asked if there are non-computable streams whose initial segment complexity is as low as a computable stream.



Solovay gave a positive answer.

Draft of a paper (or series of papers) on Chaitin's work. Unpublished notes, May 1975. 215 pages.

The world of K-trivial streams

- $\rightsquigarrow$ Computable from the halting problem i.e. Δ^0_2 (Chaitin 70s)
- $\rightsquigarrow$ Incomplete, and in fact low (Downey/Hirschfeldt/Nies/Stephan)
- $\rightsquigarrow$ Downward closed under $\leq_T$ (Hirschfeldt/Nies 2005)
- $\rightsquigarrow$ Form an ideal in the Turing degrees.

K-trivial streams in classical computability theory

Provide a 'natural' solution to Post's problem.

$$A = \{n \mid \exists e, s\ (\underbrace{W_{e,s} \cap A_s = \emptyset\ \wedge\ n > 2e\ \wedge n \in W_{e,s}}_{\text{Post's simple set}} \wedge \sum_{n<j<s} 2^{-K_s(j)} < 2^{-e})\}$$

Scott sets: Turing incomparability using the K-trivial degrees.

(Kučera and Slaman)

Cumulative hierarchy of K-trivial streams

A stream X is *K-trivial* if $K(X \upharpoonright_n) \leq K(n) + c$ for all n, some c.

K-trivial streams are stratified in a hierarchy of length ω

. . . whose c-level contains the K-trivial streams with constant c.

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Basic facts about G, by DMNY

- Computable from $\mathbf{0}^{(3)} \ldots$ i.e. Δ^0_4
- Not computable i.e. not Δ^0_1
- Not computable from the halting problem, i.e. not Δ^0_2

Is it computable from $\mathbf{0}^{(2)}$ i.e. is it Δ^0_3?

The classes of K_c-trivial streams

- They are uniformly Π^0_1 in the halting set
- The set of infinite paths through a $\mathbf{0}'$-computable tree.
- The width of these trees is computably bounded since

$$|\{\sigma \in 2^n \mid K(\sigma) \leq K(|\sigma|) + c\}| < 2^c$$

. . . by the coding theorem

Number of paths through trees of bounded width

- The number of infinite paths through a tree T with bounded width can be computed from T''.
- This is optimal!
- If a family of trees is computable from a low$_2$ oracle A then the number of paths is computable from $\mathbf{0}^{(2)}$.

Oracle A is low$_2$ if A'' is computable from $\mathbf{0}^{(2)}$; $\Sigma_2^0(A) \subseteq \Delta_3^0$.

Representing the K_c-trivial classes with simpler trees

Theorem (B. and Tom Sterkenburg)

Given a Δ^0_2 tree T which only has K_c-trivial paths we can compute the index of another Σ^0_1 tree which is K-trivial and has the same infinite paths as the original tree.

The new trees have trivial initial segment complexity.

Fact: $\mathbf{0}^{(2)}$ can compute a low$_2$ index of a K_c-trivial stream given c and the Δ^0_2 index of the stream.

Computation of $G(c)$ from $\mathbf{0}^{(2)}$

- Get the index of the original Δ^0_2 tree representing the class K_c-trivial.
- Compute the index of the K-trivial tree representing this class.
- Use $\mathbf{0}^{(2)}$ to compute a low$_2$-ness index of the new tree.
- Use $\mathbf{0}^{(2)}$ again to compute the number of infinite paths through this tree.
- This is $G(c)$

A related class: low for K streams

If a computer is given access to a powerful oracle, it will achieve better compression for many strings.

X is called low for K if $K^X = K$.

> ......*if as far as prefix-free complexity is concerned, it is not better than a computable oracle.*

This class was defined by Muchnik in 1999, who also exhibited non-computable elements in it.

Hierarchy of low for K and complexity

- Low for K streams are stratified in a cumulative hierarchy of finite classes.
- Hirschfeldt and Nies showed that they coincide with the K-trivial streams.
- Our methodology applies to this class, showing that

. . . the corresponding function giving the cardinality of the hierarchy classes is Δ^0_3.

Applications

A consequence of the main result is that $\mathbf{0}''$ can obtain the indices of the K_c-trivial strings.

This can be used to show that a number of K-related objects have lower complexity.

For example, gap functions for K-triviality.

Gap functions for K-triviality

These are non-decreasing unbounded functions f such that

$$\forall n\, [K(X \upharpoonright_n) \leq K(n) + f(n) + c] \Rightarrow X \text{ is } K\text{-trivial.}$$

- Constructed by Csima and Montalbán in 2006
- Used to obtain minimal pairs in the degrees of randomness
- Complexity: Δ^0_4
- Downey raised the question about their complexity

Complexity of gap functions

Theorem (Barmpalias/Baartse and Bienvenu/Merkle/Nies)

If f is Δ^0_2 unbounded and non-decreasing then there are uncountably many streams X such that

$$K(X\restriction_n) \leq K(n) + f(n) \text{ for all } n.$$

Theorem (Barmpalias and Martijn Baartse)

There is a Δ^0_3 gap function for K-triviality.

References

Barmpalias/Sterkenburg
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Barmpalias/Baartse
On the gap between trivial and nontrivial initial segment prefix-free complexity
Submitted.

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