# Connected Choice and the Brouwer Fixed Point Theorem

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joint work with

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*Oberwolfach, February 2012*

# Outline

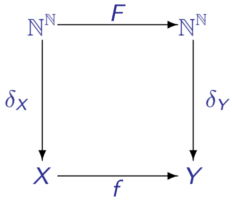
# Realizer

## Definition

A multi-valued function $f :\subseteq X \rightrightarrows Y$ on represented spaces $(X, \delta_X)$ and $(Y, \delta_Y)$ is realized by a function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ if

$$\delta_Y F(p) \in f \delta_X(p)$$

for all $p \in \mathrm{dom}(f\delta_X)$. We write $F \vdash f$ in this situation.



# Weihrauch Reducibility

**Definition (Weihrauch 1990)**

Let $f$ and $g$ be multi-valued maps on represented spaces.

- $f \leq_{\mathrm{W}} g$ ($f$ Weihrauch reducible to $g$), if there are computable functions $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for all $G$

$$G \vdash g \implies H\langle \mathrm{id}, GK \rangle \vdash f.$$

That means that there is a uniform way to transform each realizer $G$ of $g$ into a realizer $F$ of $f$ in the given way.

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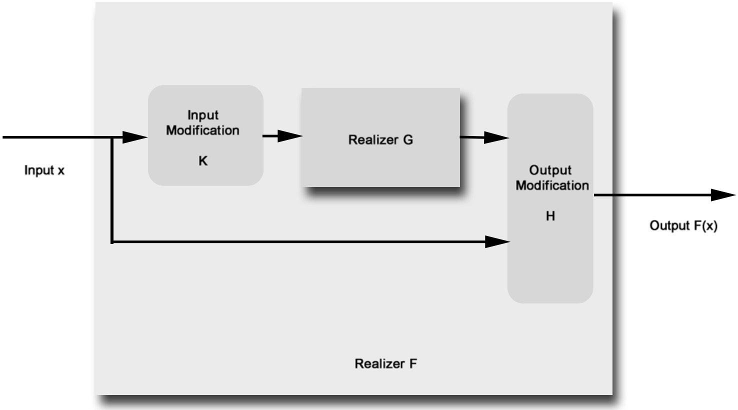


- $F(x) = H\langle x, GK(x)\rangle$ for all admissible inputs $x$.

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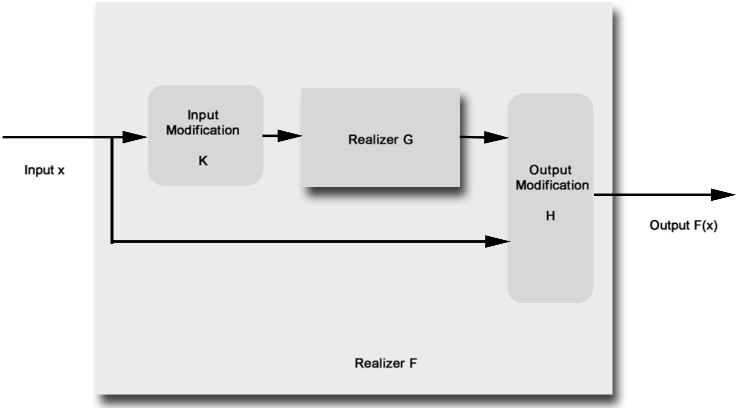


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# Algebraic Operations in the Weihrauch Lattice

**Definition**

Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq W \rightrightarrows Z$ be multi-valued maps. Then we consider the natural operations

- $f \times g :\subseteq X \times W \rightrightarrows Y \times Z$ (product)
- $f \sqcup g :\subseteq X \sqcup W \rightrightarrows Y \sqcup Z$ (coproduct)
- $f \sqcap g :\subseteq X \times W \rightrightarrows Y \sqcup Z$ (sum)
- $f^* :\subseteq X^* \rightrightarrows Y^*, f^* = \bigsqcup_{i=0}^{\infty} f^i$ (star)
- $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}, \widehat{f} = \mathsf{X}_{i=0}^{\infty} f$ (parallelization)

**Theorem (B. and Gherardi, Pauly 2009)**

*Weihrauch reducibility induces a (bounded) lattice with the sum $\sqcap$ as infimum and the coproduct $\sqcup$ as supremum and parallelization and the star operation as closure operators.*

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# Embedding of the Medvedev Lattice

## Definition

Let $A \subseteq \mathbb{N}^\mathbb{N}$.

1. By $c_A : \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}, p \mapsto A$ we denote the constant multi-valued function with value $A \subseteq \mathbb{N}^\mathbb{N}$.
2. By $\mathrm{id}|_A :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ we denote the identity restricted to $A$.

## Proposition

*Let $A, B \subseteq \mathbb{N}^\mathbb{N}$, $A \oplus B = \langle A \times B \rangle$, $A \otimes B = 0A \cup 1B$. Then*

- $A \leq_\mathrm{M} B \iff c_A \leq_\mathrm{W} c_B \iff \mathrm{id}|_B \leq_\mathrm{W} \mathrm{id}|_A$,
- $c_{A\oplus B} \equiv_\mathrm{W} c_A \times c_B \equiv_\mathrm{W} (c_A \sqcup c_B)^* \equiv_\mathrm{W} \widehat{c_A \sqcup c_B}$,
- $c_{A\otimes B} \equiv_\mathrm{W} c_A \sqcap c_B$,
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# Closure Operators and Reducibilities

## Remark

*There is a vague analogy between versions of Weihrauch reducibilities induced by closure operators and computability theoretic reducibilities:*

| **Closure operation** | **Reducibility** |
|---|---|
| $f \leq_{\mathrm{W}} g$ | *many-one reducibility* |
| $f \leq_{\mathrm{W}} g^*$ | *weak truth-table reducibility* |
| $f \leq_{\mathrm{W}} \widehat{g}$ | *Turing reducibility* |

## Question

*Can this analogy be made more precise?*

Note: we already have an embedding of Turing degrees into parallelizable Weihrauch degrees.

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# Realizing Theorems

## Definition

Any theorem $T$ of form

$$(\forall x \in X)(\exists y \in Y)\,(x \in D \Longrightarrow P(x,y))$$

is identified with $F :\subseteq X \rightrightarrows Y$ with $\mathrm{dom}(F) := D$ and

$$F(x) := \{y \in Y : P(x,y)\}.$$

## Definition (Choice)

The choice statement

$$(\forall \text{ closed } A \subseteq X)(\exists x \in X)(A \neq \emptyset \Longrightarrow x \in A)$$

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# Weak Kőnig's Lemma

**Proposition**

$\mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{C}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{C}_{[0,1]^n}$ *for all* $n \geq 1$.

Theorems that have been studied in this context include:

- Weak Kőnig's Lemma, Hahn-Banach Theorem (Gherardi, Marcone 2008)
- Intermediate Value Theorem, Baire Category Theorem, Banach's Inverse Mapping Theorem, Open Mapping Theorem, Closed Graph Theorem, Uniform Boundedness Theorem (B., Gherardi 2009)
- Nash Equilibria, Linear Inequalities (Pauly 2009)
- Hilbert's Basis Theorem (de Brecht 2010)
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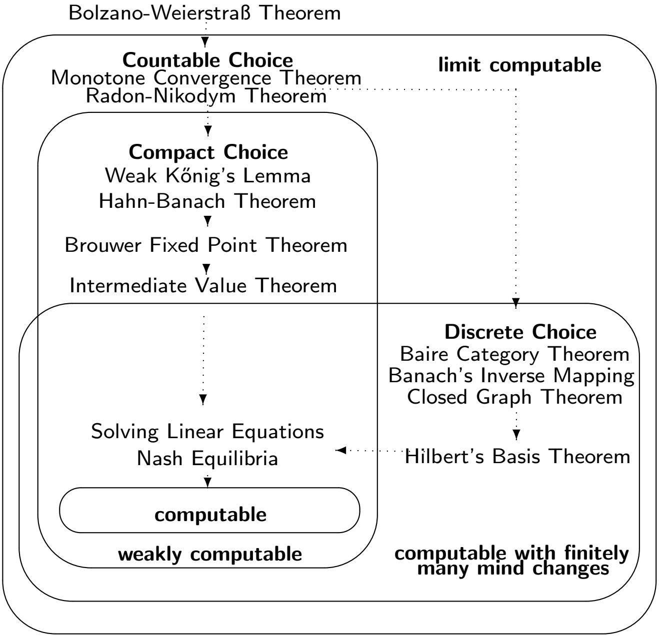
# Computable Analysis



# The Choice Operation

**Theorem (B. de Brecht and Pauly 2010)**

- $f \leq \mathsf{C}_{\{0\}} \iff f$ *is computable,*
- $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \iff f$ *comp. with finitely many mind changes,*
- $f \leq_{\mathrm{W}} \mathsf{C}_{\{0,1\}^{\mathbb{N}}} \iff f$ *is non-deterministically computable,*
- $f \leq_{\mathrm{W}} \widehat{\mathsf{C}_{\mathbb{N}}} \iff f$ *is limit computable,*
- $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \iff f$ *is Borel measurable.*

# Trees of Rational Complexes

## Definition

- A rational complex is a set $R = \{B[x_1, r_1], ..., B[x_n, r_n]\}$ of finitely many closed balls $B[x_i, r_i]$ in $\mathbb{R}^n$ with rational centers $x_i \in \mathbb{Q}^n$ and radii $r_i \in \mathbb{Q}$ such that $\bigcup R$ is connected.
- By $C\mathbb{Q}^n$ we denote the set of rational complexes.
- We write $A \Subset B$ if $A$ is compactly included in $B$, i.e. $\overline{A} \subseteq B^\circ$ for $A, B \subseteq \mathbb{R}^n$.
- A tree of rational complexes is a pair $(T, f)$ of a finitely bounded tree $T \subseteq \mathbb{N}^*$ and a function $f : T \to C\mathbb{Q}^n$ such that for all distinct $v, w \in \mathbb{N}^*$
  - $v \sqsubseteq w \Longrightarrow \bigcup f(w) \Subset \bigcup f(v)$,
  - $|v| = |w| \Longrightarrow \bigcup f(v) \cap \bigcup f(w) = \emptyset$.
- By $A_{(T,f)} := \bigcap_{n=0}^{\infty} \bigcup_{w \in T \cap \mathbb{N}^n} \bigcup f(w)$ we denote the closed set $A_{(T,f)} \subseteq \mathbb{R}^n$ represented by $(T, f)$.

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# Representation by Trees of Rational Complexes

## Proposition

*The map*

$$(T, f, b) \mapsto A_{(T,f)}$$

*that maps* $(T, f)$ *together with a bound* $b$ *to* $A_{(T,f)}$*, is computable and has a multi-valued computable right inverse, restricted to closed sets* $A \subseteq [0, 1]^n$*.*

## Corollary

*For every non-empty co-c.e. closed set* $A \subseteq [0, 1]^n$ *there is a computable sequence* $(A_i)_{i \in \mathbb{N}}$ *of bi-computable compact sets* $A_i \subseteq [-1, 2]^n$ *that is compactly decreasing, i.e.* $A_{i+1} \Subset A_i$ *for all* $i \in \mathbb{N}$ *and such that* $A = \bigcap_{i \in \mathbb{N}} A_i$*.*

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# Finding Connectedness Components

## Lemma

*Let* $(T, f)$ *be a tree of rational complexes. Then the sets*

$$C_p := \bigcap_{i=0}^{\infty} \bigcup f(p|_i)$$

*for* $p \in [T]$ *are exactly all connectedness components of* $A_{(T,f)}$.

## Definition

By $\mathrm{Con}_n : \mathcal{A}_n \rightrightarrows \mathcal{A}_n$ we denote the map with

$$\mathrm{Con}_n(A) := \{C : C \text{ is a connectedness component of } A\}$$

for every $n \geq 1$, where $\mathcal{A}_n$ dentoes the set of non-empty closed subsets $A \subseteq [0,1]^n$ represented with negative information.

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$\mathrm{Con}_n \equiv_{\mathrm{W}} \mathrm{WKL}$ *for* $n \geq 1$.

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# Computability Properties of Connectedness Components

## Corollary (le Roux, Ziegler 2009)

- *Any component of a co-c.e. closed set* $A \subseteq [0,1]^n$ *with only finitely many components is co-c.e. closed.*
- *A non-empty co-c.e. closed set* $A \subseteq [0,1]^n$ *without co-c.e. closed component has continuum many components.*

## Corollary (Open problem of le Roux, Ziegler 2009)

*There are co-c.e. closed sets* $A \subseteq [0,1]^n$ *with only countably many connectedness components one of which is not co-c.e. closed.*

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*Every non-empty co-c.e. closed set* $A \subseteq [0,1]^n$ *has a connectedness component* $C_p$ *with a low description* $p$*, in particular* $\mathsf{C}_p$ *is the set of cluster points of a computable sequence.*

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## Definition

- By $\mathcal{C}_n := \mathcal{C}([0,1]^n,[0,1]^n)$ we denote the set of continuous functions $f:[0,1]^n \to [0,1]^n$.
- By $\mathrm{BFT}_n : \mathcal{C}_n \rightrightarrows [0,1]^n$ we denote the operation defined by $\mathrm{BFT}_n(f) := \{x \in [0,1]^n : f(x) = x\}$ for $n \in \mathbb{N}$.
- By $\mathrm{CC}_n :\subseteq \mathcal{A}_n \rightrightarrows [0,1]^n$ we denote the operation defined by $\mathrm{CC}_n(A) := A$ for all non-empty connected closed $A \subseteq [0,1]^n$ and $n \in \mathbb{N}$. We call $\mathrm{CC}_n$ connected choice (of dimension $n$).

## Theorem

$\mathrm{BFT}_n \equiv_{\mathrm{W}} \mathrm{CC}_n$ *for all* $n \in \mathbb{N}$.

# The Brouwer Fixed Point Theorem

**Definition**

- By $\mathcal{C}_n := \mathcal{C}([0,1]^n, [0,1]^n)$ we denote the set of continuous functions $f : [0,1]^n \to [0,1]^n$.
- By $\mathrm{BFT}_n : \mathcal{C}_n \rightrightarrows [0,1]^n$ we denote the operation defined by $\mathrm{BFT}_n(f) := \{x \in [0,1]^n : f(x) = x\}$ for $n \in \mathbb{N}$.
- By $\mathrm{CC}_n :\subseteq \mathcal{A}_n \rightrightarrows [0,1]^n$ we denote the operation defined by $\mathrm{CC}_n(A) := A$ for all non-empty connected closed $A \subseteq [0,1]^n$ and $n \in \mathbb{N}$. We call $\mathrm{CC}_n$ connected choice (of dimension $n$).

**Theorem**

$\mathrm{BFT}_n \equiv_{\mathrm{W}} \mathrm{CC}_n$ *for all* $n \in \mathbb{N}$.

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# Connected Choice is Reducible to BFT

**Lemma**

$\mathrm{CC}_n \leq_{\mathrm{W}} \mathrm{BFT}_n$ *for all* $n$*.*

**Proof.**

- Given a connected closed set $\emptyset \neq A \subseteq [0,1]^n$ we construct a tree of rational complexes $(T, f)$ that represents $A$.
- Since $A$ is connected, there is a unique infinite path in $T$ that we can find.
- This paths yields a computable sequence $(A_i)$ of bi-computable, effectively path-connected closed sets $A_i$ that decreases compactly such that $A = \bigcap_{i=0}^{\infty} A_i$.
- We use the sequence $(A_i)$ to construct functions $g_i : [0,1]^n \to [0,1]^n$ and $f := \mathrm{id} + 2^{-4} \sum_{i=0}^{\infty} g_i$ with the property that $A$ is the set of fixed points of $f$.

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# BFT is Reducible to Connected Choice

**Lemma**

$\mathsf{BFT}_n \leq_{\mathrm{W}} \mathsf{CC}_n$ *for all* $n$.

**Proof.**

- By $\mathsf{Fix}_n : \mathcal{C}_n \to \mathcal{A}_n, f \mapsto \{x \in [0,1]^n : f(x) = x\}$ we denote the fixed point map of dimension $n$.
- We note that $\mathsf{CC}_n \circ \mathsf{Con}_n \circ \mathsf{Fix}_n(f) \subseteq \mathsf{BFT}_n(f)$.
- If we can prove that $\mathsf{Con}_n \circ \mathsf{Fix}_n$ is computable, then $\mathsf{BFT}_n \leq_{\mathrm{W}} \mathsf{CC}_n$ follows.
- Given $f$ we can compute $A = \mathsf{Fix}_n(f) = (f - \mathrm{id}_{[0,1]^n})^{-1}\{0\}$.
- Hence we can compute a tree $(T, f)$ of rational complexes representing $A$.
- Within this tree we can find an infinite path and hence a connectedness component since $\mathrm{ind}(f, R)$ is computable for rational complexes $R$ (as proved by Joe S. Miller 2002).

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# Fixability

## Theorem (Joe S. Miller 2002)

*A set $A \subseteq [0,1]^n$ is the set of fixed points of a computable function $f : [0,1]^n \to [0,1]^n$ if and only if it is a non-empty co-c.e. closed set that contains a co-c.e. closed connectedness component.*

## Proposition

$(\mathrm{Fix}_n, \mathrm{Con}_n \circ \mathrm{Fix}_n)$ *is computable and has a single-valued computable right inverse for all $n \in \mathbb{N}$.*

## Proof.

Given $(A, C)$ such that $C$ is a connectedness component of $A$ we can find $f$ such that $C = \mathrm{Fix}_n(f)$ and $g$ such that $g^{-1}\{0\} = A$. Then $h$ with

$$h(x) = (1 - g(x))x + f(x)g(x)$$

is such that $\mathrm{Fix}_n(f) = A$. □

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# Universality of Dimension Three

## Proposition

*The map*

$$A \mapsto (A \times [0,1] \times \{0\}) \cup (A \times A \times [0,1]) \cup ([0,1] \times A \times \{1\})$$

*is computable and maps* any *non-empty closed* $A \subseteq [0,1]$ *to a* connected *non-empty closed* $A \subseteq [0,1]^3$*.*

## Theorem

*For* $n \geq 3$

$$\mathsf{CC}_n \equiv_{\mathrm{W}} \mathsf{BFT}_n \equiv_{\mathrm{W}} \mathsf{WKL} \equiv \mathsf{C}_{[0,1]}.$$

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# Dimension Two

**Proposition**

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**Proposition**

$$\frac{1}{2}\mathsf{C}_{[0,1]} \leq_{\mathrm{W}} \mathsf{CC}_2 \leq_{\mathrm{W}} \mathsf{C}_{[0,1]}.$$

**Corollary (Baigger 1985, Orevkov 1963)**

- *There exists a computable function* $f : [0,1]^2 \to [0,1]^2$ *that has no computable fixed point* $x \in [0,1]^2$.
- *There exists a non-empty connected co-c.e. closed subset* $A \subseteq [0,1]^2$ *without computable point.*

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# Aspects of Dimension

## Proposition

$\mathrm{CC}_0 <_{\mathrm{W}} \mathrm{CC}_1 <_{\mathrm{W}} \mathrm{CC}_2 \leq_{\mathrm{W}} \mathrm{CC}_3 \equiv_{\mathrm{W}} \mathrm{CC}_n$ *for all* $n \geq 3$.

- $\mathrm{CC}_0$ is computable.
- $\mathrm{CC}_1 \equiv_{\mathrm{W}} \mathrm{BFT}_1 \equiv_{\mathrm{W}} \mathrm{IVT}$ is non-uniformly computable, but not uniformly computable.
- $\mathrm{CC}_2$ is not non-uniformly computable (by the Baigger/Orevkov example).
- $\mathrm{CC}_n$ is computably complete (equivalent to WKL) for $n \geq 3$.

## Conjecture

$\mathrm{CC}_2 <_{\mathrm{W}} \mathrm{CC}_3$.

## Theorem (Idempotency)

$\mathrm{CC}_1 <_{\mathrm{W}} \mathrm{CC}_1 \times \mathrm{CC}_1 <_{\mathrm{W}} \mathrm{CC}_2$.

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