Problems of autostability relative to strong constructivizations

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Constructive models by A.I.Malcev, 1961

A. I. Maltsev gave start to systematic investigation of constructive models on the base of numberings or naming of elements of models and start to study algorithmic properties of structures on the base of classical theory of algorithms.

Let $\mathfrak{M}$ – be a model with signature σ. If ν is enumeration of the main set of model $\mathfrak{M}$, then we call the pair $(\mathfrak{M}, \nu)$ a numbered model. Here we take as value of every constant a_i an element $\nu(i)$ for each $i \in \mathbb{N}$. Let $D(\mathfrak{M}, \nu)$ be the quantifier-free theory of model $\mathfrak{M}_\nu$, i. e. the set of sentences without quantifier in signature $\sigma_\mathbb{N}$, which are true in model $\mathfrak{M}_\nu$.

Definition

The numbered model $(\mathfrak{M}, \nu)$ is constructive if the set $D(\mathfrak{M}, \nu)$ is recursive, i. e. there exists an algorithm for testing validity for quantifier-free formulas on elements of this model.

A.I.Maltsev

Recursive equivalence of constructive models by A.I.Malcev

In connection with problem of uniqueness of constructive enumeration for a given model A. I. Maltsev introduced the notion of recursively stable model. He noticed that finitely generated algebraic systems are recursively stable.
Let $(\mathfrak{M}, \nu)$ and $(\mathfrak{M}, \mu)$ – be two numbered models for the model $\mathfrak{M}$.

Definition

The numberings ν and μ of model $\mathfrak{M}$ are *recursively equivalent*, if there exist recursive functions f and g such that $\nu = \mu f$ and $\mu = \nu g$.

Recursive equivalence of constructive models by A.I.Malcev

Theorem

Two constructivizations ν and μ of model $\mathfrak{M}$ are recursively equivalent, if for any subsets $X \subseteq M^k, k \geq 1$ the set $\nu^{-1}(X)$ is recursive iff the set $\mu^{-1}(X)$ is recursive

Autoequivalence of constructive models by A.I.Malcev

In his paper "Recursive groups" A. I. Maltsev has introduced the notion of autoequivalence, which is weaker.

Definition

Two constructivizations ν and μ of the model $\mathfrak{M}$ are *autoequivalent*, if there exist recursive function f and automorphism λ of the model $\mathfrak{M}$ such that $\lambda\nu = \mu f$.

The model is called *autostable* if for every two constructivizations of the model $\mathfrak{M}$ are autoequivalent.
A. I. Maltsev has shown that for infinite dimensional vector space over the field of rational numbers one can construct two different constructivizations.
The question on connections between autostability and model-theoretic properties belongs to studies of the same problem.

For the first time the problem of autostability for algebraically closed fields was stated in somewhat different language by B. L. van der Waerden. A. Fröhlich and J. C. Shepherdson have answered this question negatively.
They have shown that in some cases there is no algorithm for construction of computable isomorphism between algebraic closures of a field that are constructed in different ways. This paper gave start to systematic study of the problem of study different computable representations for algebraic systems.

Strongly constructive models by Yu.L.Ershov, 1968

Let $Th(\mathfrak{M}, \nu)$ be the elementary theory of model $\mathfrak{M}_\nu$, i. e. the set of sentences in signature $\sigma_\mathbb{N}$, which are true in model $\mathfrak{M}_\nu$.

Definition

A numbered model $(\mathfrak{M}, \nu)$ is called strongly constructive, if elementary theory $Th(\mathfrak{M}, \nu)$ in signature $\sigma_\mathbb{N}$ is decidable.

Yu.L.Ershov proved that any decidable theory has a strong constructive model and started to study the model theory for decidable models and starts to construct Model theory of strongly constructive models.

Decidable models by M.Morley, 1971

M. Morley has introduced an equivalent notion of decidable model.
He solved the problem about decidability for countable saturated models.

It is evident that every strongly constructive model is also constructive one, but the opposite is not true. Notice also that elementary theory of strongly constructive model is decidable.

Theorem

The constructivizations ν and μ of the model $\mathfrak{M}$ are autoequivalent, if for any $n \geq 1$ and any subset $S \subseteq M^n$ the set of sequences of ν-numbers of elements from S is computable iff there exists an automorphism λ of our model $\mathfrak{M}$ such that the set of sequences of μ-numbers of elements from $\lambda(S)$ is computable.

Let Δ be a class of functions such that Δ is closed relative to superposition and for any permutation f of N from Δ the function f^{-1} from Δ too.

The constructivizations ν and μ of the model $\mathfrak{M}$ are Δ-*autoequivalent* relative to strong constructivization, if there exist function f from Δ and automorphism λ of the model $\mathfrak{M}$ such that $\lambda\nu = \mu f$.
The model is called Δ-*autostable* relative to strong constructivization if for every two strong constructivizations ν_1 and ν_2 of the model $\mathfrak{M}$ there exist automorphism α of model $\mathfrak{M}$ and function f from Δ such that $\alpha\nu_1 = \nu_2 f$.

In the series of papers S. Goncharov, J. Knight, V. Harizanov and our colleagues we have research the problems of Δ-autostability relative to Δ where Δ are different classes of hyperarithmetical hierarchy, B. Khoussainov, F. Stephan with coauthors start to study Δ-autostability relative to Δ where Δ are different classes of Ershov's hierarchy.

We can consider a Turing degree a and define Δ_a as a class of all function recursive relative to this degree a. In this case we have Δ_a-autostability. If there exists a smallest degree a such that the model $\mathfrak{M}$ is Δ_a-autostable then we call the model $\mathfrak{M}$ a-autostable.
E. Fokina, I. Kallimulin and R. Miller gave some partial answer on my question about Turing degrees of autostability. I gave answer about Turing degrees of autostability relative to strong constructivizations for the case almost prime models. It is exactly all c.e. Turing degrees.

Problem

Is there a prime model strongly constructivizable but without Turing degrees of autostability relative to strong constructivizations.

Index Set

Definition

The **index set** of a structure $\mathcal{A}$ is the set $I(\mathcal{A})$ of all indices of computable (isomorphic) copies of $\mathcal{A}$, where a computable index for a structure $\mathcal{B}$ is a number e, such that $\varphi_e = \chi_{D(\mathcal{B})}$.

Definition

For a class K of structures, closed under isomorphism, the **index set** is the set $I(K)$ of all indices for computable members of K.

$$I(K) = \{e : \exists \mathcal{B} \in K \, \varphi_e = \chi_{D(\mathcal{B})}\}$$

Index Set

Problem

. To study complexity of Index sets of Δ-autostable models for different sets Δ and connection between this Index sets.

Some definitions

Recall that theory is called *countably categorical*, if it is complete and has countable model, which is unique up to isomorphism. The theory is called *Ehrenfeucht theory* if it is complete and has finite number of countable models but is not countably categorical.

If we enrich the model $\mathfrak{M}$ by adding constants to its signature for finite collection A of elements from $\mathfrak{M}$, we call *finite enrichment of model* $\mathfrak{M}$ *by constants* the resulting model and denote it $(\mathfrak{M}, \bar{a})_{a \in A}$, where $\bar{a}$ is finite collection of elements from $\mathfrak{M}$.

R. Vaught gave a characterization of smallest relative to elementary embedding model. These models play very important role in the model theory.
We call $\mathfrak{M}$ *prime model* of complete theory T, if it can be elementary embedded into every other model of theory T.

The model $\mathfrak{M}$ is atomic if for every collection of elements $a_1, \dots, a_n \in |\mathfrak{M}|$ there exists formula $\psi(x_1, \dots, x_n)$ such that $\mathfrak{M} \models \psi(a_1, \dots, a_n)$ and for every formula $\varphi(x_1, \dots, x_n)$, if $\mathfrak{M} \models \varphi(a_1, \dots, a_n)$, then

$$\mathfrak{M} \models (\forall x_1) \dots (\forall x_n)\big(\psi(x_1, \dots, x_n) \to \varphi(x_1, \dots, x_n)\big)$$

.
Such formula $\psi(x_1, \dots, x_n)$ is called complete formula for theory of this model.

Theorem (R. Vaught)

A model of complete theory is prime iff this model is atomic.

By Vaught criterion countable model is prime if and only if it is atomic.
Let us call a model *almost prime*, if it is prime in some finite enrichment with constants.

Autostability relative to strong constructivizations in complete decidable theories

I would like to tell about some investigations of existence of complete decidable theories with autostable prime model in finite enrichment with constants, but with prime model, which is not autostable relative to strong constructivizations. We also investigate the existence of decidable complete theories with prime models, which are autostable relative to strong constructivizations, but for such theories exist models, which are not autostable relative to strong constructivizations, but are prime in finite enrichment with constants.

R. L. Vaught has shown that the Ehrenfeucht theory with two countable models is impossible. But for every $n \geq 3$ there exists Ehrenfeucht theory T with n countable models.
M. G. Peretyatkin has shown that any decidable Ehrenfeucht theory has computable family of recursive main types and only its prime model is decidable.
He proved that for any decidable Ehrenfeucht theory the prime model of this theory is decidable.

Problems of existence of constructivizations for models, their autostability and algorithmic dimension are central for mainstream investigations in the theory of computable models. We will consider here the problem of autostability relative to strong constructivizations.

A. T. Nurtazin criterion, 1973

We begin with an important theorem by A. T. Nurtazin.
A. T. Nurtazin has found criteria for autostability for the case of autostability relative to strong constructivizations. This criteria shows strong connections between the problem of autostability relative to strong constructivizations and the properties of model.

Theorem (A. T. Nurtazin criterion)

Let $\mathfrak{M}$ – be strongly constructive model of complete theory T. Then following conditions are equivalent:

- *1) $\mathfrak{M}$ is autostable relative to strong constructivizations;*
- *2) there exists finite sequence $\bar{a}$ of elements from M such that enrichment $(\mathfrak{M}, \bar{a})$ of model $\mathfrak{M}$ with this constants is prime model and collection of sets of atoms of computable boolean Lindenbaum algebras $F_n\big(Th(\mathfrak{M}, \bar{a})\big)$ of theory $Th(\mathfrak{M}, \bar{a})$ over the set of formulas with n variables is uniformly computable.*

Theorem (Marchuk, Fokina)

The index set $IndD_{CatD}$ of strongly computable models and autostable relative to strong constructivizations has a complexity exactly Σ^0_3.

Theorem (Gonchrov-Marchuk)

The index set $IndCom_{CatD}$ of computable models and autostable relative to strong constructivizations is $\Sigma_{\omega+3}$.

E.Palyutin example, 1974

It is easy to get for any $n \geq 3$ an example of decidable Ehrenfeucht theory with n countable models but prime model is not autostable relative to strong constructivizations. Such example is based on the same theorem by A. T. Nurtazin and Palyutin's example.

E. A. Palyutin has proposed an example of countably categorical theory with following algorithmic properties.

Theorem (E. A. Palyutin)

There exists decidable countably categorical theory T with elimination of quantifiers for which function $\alpha_n(T)$, giving cardinality of Lindenbaum algebra of theory $F_n(T)$ over the set of formulas with n free variables is not general recursive.

Autostability relative to strong constructivizations in countable categorical theory

Function $\alpha_n(T)$ is called the function of Ryll-Nardzewsky for countably categorical theory T. From nonrecursiveness of Ryll-Nardzewsky function in the theory of E. A. Palyutin it follows that collection of complete formulas of this theory is not computable.

Corollary

There exists countably categorical theory with strongly constructivizable countable model, which is not autostable relative to strong constructivizations and is even not autostable.

For decidable theories there always exists strongly constructivizable model due to theorem by Yu. L. Ershov. In the case of Ehrenfeucht theories there exists more powerful result.

Theorem (M. G. Peretyatkin)

The prime model of decidable Ehrenfeucht theory is strongly constructivizable.

In general case was produced the following criterion for decidability of prime models.

Theorem (S. S. Goncharov, 1973 and L. Harrington, 1975)

A prime model of decidable theory is strongly constructivizable if and only if the family of all principal types of this theory is computable.

D.Hirshcfeldt noted that example of decidable theory with not-decidable prime model and all types of which is computable gave us example of models without Turing degree and spectrum of this model is exactly all non-computable Turing degrees and spectrum of degrees of strong constructivization is the same.

Autostability relative to strong constructivizations in Ehrenfeucht theories

Theorem

For any $n \geq 3$ there exists decidable Ehrenfeucht theory T_n with elimination of quantifiers and exactly n countable models such that its countable models are not autostable relative to strong constructivizations and not autostable. Moreover, the class of constructivizations up to recursive isomorphism for each its model is not computable.

Autostability relative to strong constructivizations in Ehrenfeucht theories for almost prime models upward

Theorem

There exists Ehrenfeucht theory with $n \geq 3$ countable models, which is autostable relative to strong constructivizations and even autostable, but it also has a model, which is prime in finite enrichment with constants and strongly constructivizable, but not autostable.

Autostability relative to strong constructivizations in theories upward

K. Zh. Kudaibergenov constructed a theory, possessing a prime model not autostable relative to strong constructivizations but also possessing some almost prime model, which is autostable relative to strong constructivizations and has a strong constructivizations.
But in his theory there exists uncountable set of types.

Theorem (K. Zh. Kudaibergenov, 1984)

There exists complete decidable theory with uncountable set of types possessing a prime model, which has a strong constructivization and is not autostable relative to strong constructivizations; but for this theory there exists a prime model in finite enrichment with constants such that it has a strong constructivization and is autostable relative to strong constructivizations.

Autostability relative to strong constructivizations in ω-stable theories

Theorem

The decidable theory is not ω-stable if for this theory there exists a prime model in finite enrichment with constants such that it has a strong constructivization and is autostable relative to strong constructivizations but prime model is decidable and is not autostable relative to strong constructivizations.

Autostability relative to strong constructivizations in uncountably categorical theories down

Corollary

There can not exist an uncountably categorical theory with prime model, which is not autostable relative to strong constructivizations but with some other model, which is autostable relative to strong constructivizations.

Corollary

If some model for uncountably categorical theory is autostable relative to strong constructivizations then its prime model will also be autostable relative to strong constructivizations.

Theorem

For any n there exists decidable uncountably categorical theory such that all of its countable models are strongly

Autostability relative to strong constructivizations in theories with countable types down

Theorem

2010 There exists complete decidable theory with countably many countable models, with strongly constructivizable prime model, which is not autostable relative to strong constructivizations but with strongly constructivizable prime model in finite enrichment with constants that is autostable relative to strong constructivizations.

Algebraically closed fields of characteristic 0 provide example of uncountably categoric theory for which all countable models with exception of countably saturated one will be autostable relative to strong constructivizations. We shall construct now one more example of uncountably categoric theories.

Theorem

2010

There exists uncountably categorical theory with prime model, which is autostable relative to strong constructivizations and all other models are not autostable relative to strong constructivizations.

Autostability relative to strong constructivizations in Ehrenfeucht theories down

Theorem (Main theorem, 2011)

For any $n \geq 5$ the existence of Ehrenfeucht theory with n countable models and the prime model of this theory is not autostable relative to strong constructivizations but some almost prime model of this theory is strongly constructivizable and autostable relative to strong constructivizations.