Generic computability and asymptotic density

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References

- Generic case complexity, decision problems in group theory and random walks, (Kapovich, Miasnikov, Schupp and Shpilrain) J. Algebra, (2003).
- Generic computability, Turing degrees and asymptotic density (Jockusch and Schupp), to appear in *Journal of the London Mathematical Society*
- Asymptotic density and computably enumerable sets (Downey, Jockusch and Schupp), in preparation.

The first paper introduced time-bounded generic computability and applied it to decision problems in group theory. The second studies generic computability in the context of classical computability theory. The third concentrates on asymptotic density in this context. Today, I will mainly discuss the third paper.

Results are joint with Downey and Schupp, except as stated.

(Asymptotic) density of subsets of ω

Definition

Let $A \subseteq \omega$, $n \in \omega$. Identify n and $\{0, 1, \ldots, n-1\}$.

1. $\rho_n(A) = |A \cap n|/n$ (density of A up to n)
2. $\rho(A) = \lim_n \rho_n(A)$, provided the limit exists. (Density of A.)

Examples

ρ(*multiples of n*) = $1/n$

ρ(*squares*) $= 0$

ρ(*primes*) = 0

If A is 1-random, then $\rho(A) = 1/2$

ρ(*square-free numbers*) $= 6/\pi^2$

Generic computability for subsets of ω

Definition

Let $A \subseteq \omega$. Then A is *generically computable* if there is a partial computable function ψ such that:

- $\psi(n) = A(n)$ for *all* n in the domain of ψ
- The domain of ψ has density 1.

Intuitively, A is generically computable if there is a partial algorithm for computing A which never lies and which answers very frequently.

Proposition

(JS) A is generically computable if and only if there are c.e. sets $U \subseteq A, V \subseteq \overline{A}$ such that $\rho(U \cup V) = 1$.

Examples of generic computability and noncomputability

Every c.e. set of density 1 is generically computable. Hence, every maximal set is generically computable.

Every Turing degree contains a generically computable set.

(JS) Every nonzero Turing degree contains a set which is not generically computable.

No bi-immune set is generically computable. Hence, no 1-generic set is generically computable, and no 1-random set is generically computable.

A basic question

Suppose the notion of generic computability is modified so that the partial function ψ must have a computable domain.

Question

Would the same sets be generically computable?

The answer is “yes” if and only if every c.e. set of density 1 has a computable subset of density 1.

Upper and lower density

Definition

Let $A \subseteq \omega$.

1. The *upper density* of A, denoted $\overline{\rho}(A)$, is $\limsup_n \rho_n(A)$.
2. The *lower density* of A, denoted $\underline{\rho}(A)$, is $\liminf_n \rho_n(A)$.

Example

Every 1-generic set has upper density 1 and lower density 0. Hence, no 1-generic set has a density.

Computable subsets with large upper density

Theorem

(Barzdin, 1970). Let A be a c.e. set and let $\epsilon > 0$ be a real number. Then A has a computable subset B such that $\overline{\rho}(B) \geq \overline{\rho}(A) - \epsilon$.

Idea of proof: Choose a rational number q such that $\overline{\rho}(A) - \epsilon < q < \overline{\rho}(A)$. Then $\rho_n(A) \geq q$ for infinitely many n. Seek and ye shall find.

Theorem

Let A be a c.e. set such that $\overline{\rho}(A)$ is a Δ^0_2 real. Then A has a computable subset B with $\overline{\rho}(B) = \overline{\rho}(A)$.

Idea of proof: If $\overline{\rho}(A)$ **is** a rational number r, do the previous proof over all q of form $r - 2^{-n}$. In general, $\overline{\rho}(A)$ is the limit of a computable sequence of rational numbers, and this suffices.

Pushing up lower density

Theorem

Let A be a c.e. set, and suppose $\epsilon > 0$ is a real number. Then A has a computable subset B such that $\underline{\rho}(B) \geq \underline{\rho}(A) - \epsilon$.

Idea of proof: Let q be a rational number such that

$$\underline{\rho}(A) - \epsilon < q < \underline{\rho}(A)$$

Fix n_0 so that $\rho_n(A) \geq q$ for all $n \geq n_0$.
Seek far ahead, and ye shall still find.

A bogus conjecture

"Conjecture" Every c.e. set of density 1 has a computable subset of density 1.

Idea for "proof": Do the previous argument over all q of form $1 - 2^{-k}$.

A BOGUS CONJECTURE

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Idea for "proof": Do the previous argument over all q of form $1 - 2^{-k}$.

The error: Given a rational $q < 1$, we may choose n_0 such that $\rho_n(A) \geq q$ for all $n \geq n_0$. However, n_0 may not depend effectively on q.

Rescuing a theorem

Definition

- The function w *witnesses* that the set A has density 1 if $(\forall k)(\forall n \geq w(k))[\rho_n(A) \geq 1 - 2^{-k}]$.
- The set A has density 1 *effectively* if there is a computable function which witnesses that A has density 1.

Theorem

If A is a c.e. set which has density 1 effectively, then A has a computable subset B which has density 1 effectively.

Idea of proof : The previous bogus proof is now OK.

Δ^0_2 witness functions

Theorem

Let A be a c.e. set of density 1. TFAE:

1. *A has a computable subset of density 1*
2. *A has a Δ^0_2 witness function*

Idea of proof. To show that (1) implies (2), note that every set B of density 1 has a witness function $w \leq_T B'$.
To show that (2) implies (1), use the same idea as to show that every c.e. set of effective density 1 has a computable subset of effective density 1. In place of a witness function, use a computable approximation to a witness function.

A counterexample, at last

Theorem

(JS) There is a c.e. set of density 1 with no co-c.e. subset of density 1.

Idea of proof. Define

$$R_n = \{k : 2^n \mid k \quad \& \quad 2^{n+1} \nmid k\}$$

The sets R_n are pairwise disjoint, uniformly computable, and have positive density. Requirements:

$$P_e : R_e \subseteq^* A$$

$$N_e : \text{ If } W_e \cup A = \omega \text{ then } \overline{\rho}(W_e) > 0$$

The P_e's imply that $\rho(A) = 1$.
The N_e's imply that A has no co-c.e. subset of density 1.

Strategy for meeting P_e and N_e, affecting A only on R_e.

1. Choose an interval $I_0 \subseteq R_e$ which is currently disjoint from A and "large" in the sense that $\rho_m(I_0) \geq (1/2)\rho_m(R_e)$, where $m = \max I_0 + 1$. Restrain all elements of I_0 from entering A.
2. Wait for a stage s_0 at which W_e covers I_0. If s_0 never occurs, we win because $W_e \cup A \neq \omega$.
3. At stage s_0, dump I_0 into A, and start over by looking for a new large interval I_1, etc.
4. To meet P_e, if $s \in R_e$, put s into A at s if s is not restrained.

If there are infinitely many cycles, we win because $\overline{\rho}(W_e) > 0$.

A stronger counterexample

Theorem

There is a c.e. set of density 1 *with no computable subset of nonzero density.*

This is a bit surprising because every c.e. set of density 1 has a computable subset which has upper density 1 and lower density as close to 1 as desired.

Degrees of counterexamples

Theorem

Let **a** *be a c.e. degree. TFAE:*

1. **a** *is not low.*
2. *There is a c.e. set of degree* **a** *which is of density* 1 *but has no computable subset of density* 1*.*
3. *There is a c.e. set of degree* **a** *which is of density* 1 *but has no computable subset of nonzero density.*

We already know that (2) implies (1).

Sketch of proof that (1) implies (2)

Given a nonlow c.e. set C, we must construct $A \leq_T C$ such that A has density 1 and no computable subset of A has density 1.

To make $A \leq_T C$ use ordinary permitting, modified so that s itself can be enumerated into A at stage s without permission. This is done automatically if s is not restrained. We make A have density 1 as before.

As before, let N_e be the statement:

$$N_e : W_e \cup A = \omega \Rightarrow \overline{\rho}(W_e) > 0$$

We will define a computable function $g(e, i, s)$. Let $L_{e,i}$ be the statement:

$$\lim_s g(e, i, s) = C'(i)$$

Use $R_{e,i} := R_{\langle e,i \rangle}$ to meet the **requirement**

$$N_{e,i} : N_e \text{ or } L_{e,i}$$

Suppose all requirements $N_{e,i}$ are met. If N_e is not met, then all $L_{e,i}$ hold and C is low, a contradiction. Hence, if suffices to meet $N_{e,i}$. We set $g(e,i,0) = 0$. Unless otherwise indicated, we $g(e,i,s+1) = g(e,i,s)$.

Strategy to meet $N_{e,i}$ and $P_{\langle e,i \rangle}$:

1. Wait for a stage s_0 with $i \in C'[s_0]$. If s_0 never occurs, we win via $\lim_s g(e,i,s) = 0 = C'(i)$ and $R_{e,i} \subseteq A$.
2. At stage s_0, let u_0 be the use of the computation $i \in C'$. Choose a large interval $I_0 \subseteq R_{e,i}$ with $u_0 < \min I_0$, with I_0 currently disjoint from A. Restrain elements of I_0 from entering A. Wait for one of the following to occur:
(a) C changes below u_0 or
(b) W_e covers I_0
If (a) occurs, cancel I_0 and dump it into A (which is permitted). Drop all restraint and start over, waiting for s_2 with $i \in C'[s_2]$, etc.
If (b) occurs, say at s_1, set $g(e,i,s_1) = 1$. Then start waiting for (a) to occur, and when it does, dump and restart as above.

If there is an infinite wait, we win. Suppose there is no infinite wait.
If (b) occurs in infinitely many cycles, we win via $\overline{\rho}(W_e) > 0$.
Otherwise, we win via $\lim_s g(e, i, s) = 0 = C'(i)$.
We meet $P_{\langle e,i \rangle}$ as before.

More on low c.e. sets

Theorem

(JS) The densities of the computable sets are exactly the Δ^0_2 reals in the interval $[0, 1]$.

Theorem

Let A be a low c.e. set of density d and let d_0 be a Δ^0_2 real in the interval $[0, d]$. Then A has a computable subset B of density d_0.

SUMMARY For low c.e. degrees the situation is as good as possible. For nonlow c.e. degrees the situation is as bad as possible.

Absolute undecidability

Consider now the extreme opposite of generic computability.

Definition

(Miasnikov and Rybalov) A set A is *absolutely undecidable* if every partial computable function which agrees with A on its domain has a domain of density 0.

Proposition

A is absolutely undecidable if and only if every c.e. subset of A and of $\overline{A}$ has density 0.

Example

Every bi-immune set is absolutely undecidable. Hence, every 1-generic set and every 1-random set is absolutely undecidable.

An Open Problem

Recall: Every nonzero Turing degree contains a set which is not generically computable.

Question

Does every nonzero Turing degree contain a set which is absolutely undecidable?

A partial result towards a negative answer:

Theorem

There is a noncomputable set A such that for every $B \leq_T A$ either $\overline{B}$ has an infinite c.e. subset or B has a c.e. subset of positive upper density.

This extends the result that there is a nonzero degree with no bi-immune set. It is proved by modifying a new proof of this result.

Recall:

THEOREM

(JS) Let r be a real number in the interval $[0,1]$. The following are equivalent:

1. *r is the density of some computable set*
2. *r is the limit of a computable sequence of rational numbers*

THEOREM

Let r be a real number in the interval $[0,1]$. Then the following are equivalent:

1. *r is the density of a c.e. set*
2. *There is an effective double sequence of rational numbers $\{q_{i,s}\}_{i,s\in\omega}$ such that $q_{i,s} \leq q_{i,s+1}$ for all i and s, for all i there are only finitely many s with $q_{i,s} \neq q_{i,s+1}$, and $\lim_i \lim_s q_{i,s} = r$.*

Theorem

Let r be a real number in the interval $[0,1]$.

1. *r is the lower density of a computable set if and only if r is left Σ^0_2*
2. *r is the upper density of a computable set if and only if r is left Π^0_2*

Theorem

Let r be a real number in the interval $[0,1]$.

1. *r is the density of a c.e. set if and only if r is left Π^0_2*
2. *r is the lower density of a c.e. set if and only if r is left Σ^0_3*
3. *r is the upper density of a c.e. set if and only if r is left Π^0_2*

Corollary

There is a real number which is the density of a c.e. set but not of any computable set. (ETC)

Relativization and minimal pairs

Recall that the set C is *generically computable* if there is a partial computable function ψ such that $\psi(n) = A(n)$ for all n in the domain D of ψ, and D has density 1. This notion can be relativized in the obvious way.

Definition

We say that (A, B) is a *minimal pair for relative generic computability* if A and B are not computable, and every set C which is generically computable relative to both A and B is generically computable.

A recent surprising result:

Theorem

(Greg Igusa) There does not exist a minimal pair for relative generic computability.

Generic reducibility

Note that relative genericity is not transitive.

Definition

- A partial function ψ is a *generic description* of a set A if $\psi(n) = A(n)$ for all n in the domain of ψ, and the domain of ψ has density 1. We identify ψ with $\{\langle x, y\rangle : \psi(x) = y\}$
- $B \leq_g A$ if there is an enumeration operator which maps any generic description of A to a generic description of B

Then $\leq_g$ is transitive. The corresponding degrees are called *generic degrees*.

Question

Is there a minimal pair of generic degrees?