Properties of limitwise monotonicity spectra of Σ^0_2 sets

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X-c.e. families

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- (Yates). $\mathbf{SpE}(\mathcal{COINFCE}) = \{\mathbf{x} : \mathbf{0}''' \leq \mathbf{x}''\}$.

Typical examples I

- (Wehner). Let $\mathcal{W} = \{\{n\} \oplus F : F \text{ is finite } \& \ F \neq W_n\}$. Then $\mathbf{SpE}\,(\mathcal{W}) =$ the non-zero degrees.

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Corollary. 1. (Slaman, Wehner) There are non-computable structures which are computable in every nonzero degree
2. There are computable non-decidable structures which are decidable in every nonzero degree

Typical examples II

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Typical examples III

- If $A \in \mathbf{a}$ is low then for $\mathcal{U} = \{\{n\} \oplus F : F \text{ is finite } \& \ F \neq W_n^A\}$. we have $\mathbf{SpE}(\mathcal{U}) = \{\mathbf{x} : \mathbf{x}' \not\leq_m \mathbf{a}'\} =$ not below $\mathbf{a}$.

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- In general, if $\mathcal{C}$ is a class with a Δ_2^0-enumeration closed downward under m-reducibility then there is a Δ_2^0-enumeration $\{U_n\}_{n\in\omega}$ of the class $\mathcal{C}$ such that

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- In particular,

$$\mathbf{Sp}(\{\{n\} \oplus F : F \text{ is finite } \& \ F \neq U_n\}) = \text{non-superlow}$$

for a Δ_2^0-enumeration $\{U_n\}_{n\in\omega}$ of all ω-c.e. sets.

Typical examples IV

- If $A \in \mathbf{a}$ is c.e. then for

$$\mathcal{U} = \{\{n\} \oplus F : F \text{ is a range of a 1-1 p.r.f. \& } F \neq W_n^A\}.$$

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Typical examples V

- If $\{X_n'\}_{n\in\omega}$ is a Δ_2^0-sequence such that $X_n >_T \emptyset$ and $\deg(X_n) \cap \deg(X_m) = \mathbf{0}$, $n \neq m$, then for

$$\mathcal{U} = \{\{n\} \oplus F : F \text{ is finite } \& \ F \neq W_n^{X_n}\}.$$

we have $\mathbf{SpE}\,(\mathcal{U})$ = non-zero, but NOT uniformly.

Corollary. There are families $\mathcal{W}$ and $\mathcal{U}$ of finite sets with the same $\mathbf{SpE}$, such that an enumeration of $\mathcal{U}$ can not be derived uniformly from arbitrary enumeration of $\mathcal{W}$.

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Problem. Are 1,2 and 3 enough?
Answer. No.

Why the answer is No for co-meager spectra

Theorem. (Greenberg, Montalban, Slaman). There is a family $\mathcal{U}$ such that $\mathbf{SpE}(\mathcal{U})$ is co-null and meager.

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- Given a sequence of infinite sets $\mathcal{S} = \{S_n\}_{n\in\omega}$. Then for the family $\mathcal{LM}(\mathcal{S}) = \{\{n\} \oplus \omega \upharpoonright m : m \in S_n\}$ conditions 1,2 and 3 are equivalent to $\mathcal{S}$ is uniformly Σ^0_2.

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More typical examples

- (K, Khoussainov, Melnikov). Let $\mathcal{S} = \{S_n\}_{n\in\omega}$ be such that

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Then

$$\mathcal{LM}(\mathcal{S}) = \{\{n\} \oplus \omega \restriction m : m \neq |W_n|\}$$

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- (Wallbaum). There are exists a set S such that $\mathbf{SpE}\,(\mathcal{LM}(\mathcal{S}))$ is co-null (in fact, it contains all 2-random) and $\mathbf{SpE}\,(\mathcal{LM}(\mathcal{S})) \subseteq$ non-zero.

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Corollary (Greenberg, Montalban, Slaman) There is a structure (indeed an abelian p-group) whose degree spectrum is co-meager and null.

Open questions I

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  $\mathbf{SpE}(\mathcal{U}) = ...?$

Open questions II

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- Let $\mathbf{SpE}(\mathcal{LM}(S_1)) \subseteq \mathbf{SpE}(\mathcal{LM}(S_2))$. Does there exists an effective procedure getting an enumeration of $\mathcal{LM}(S_1))$ given any enumeration of $\mathcal{LM}(S_2))$?