An example related to a theorem of John Gregory

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Jon Barwise and the unity of computability, set theory, and model theory

Gregory's Theorem is from 1970. At that time, parts of model theory (infinitary logic), set theory (fine structure of L), and computability (α-recursion) were closely related. Jon Barwise was a great spokesperson for the unity of logic, but there were many contributors to the body of work, including Ronald Jensen, Carole Karp, Georg Kreisel, Saul Kripke, Richard Platek, John Schlipf, Jean-Pierre Ressayre, and Gerald Sacks and some of his students, in particular, Richard Shore and Sy Friedman.

Background on infnintary logic

For a predicate language L, the infinitary formulas of $L_{\omega_1\omega}$ involve countable disjunctions and conjunctions, but only finite sequences of quantifiers. We restrict to formulas with only finitely many free variables.

For an $L_{\omega_1\omega}$ formula, we cannot in general bring the quantifiers "outside", as in prenex normal form, but we can bring the negations "inside". The result is another kind of normal form.

We classify formulas in normal form as Σ_α or Π_α for countable ordinals α.

Admissible sets and admissible fragments

- An *admissible* set is a transitive model A of *Kripke-Platek* set theory, a weakening of ZF in which the power set axiom is dropped and separation and collection are restricted to bounded formulas. Some people also drop infinity.
- For a countable admissible set A, the fragment L_A consists of the $L_{\omega_1 \omega}$ formulas in A,
- X is Σ_1 *on* A if X is defined in $(A, \in)$ by a formula with only existential and bounded quantifiers,
- X is *A-finite* if $X \in A$.

We write $\mathcal{M} \preceq_{L_A} \mathcal{N}$ if satisfaction of L_A formulas by tuples from $\mathcal{M}$ is the same in $\mathcal{M}$ and $\mathcal{N}$. We use $\prec$ instead of $\preceq$ if the extension is proper.

The least admissible set

Let $A = L_{\omega_1^{CK}}$. This is the least admissible set (with ω). For this A,

- L_A-formulas are essentially the computable infinitary formulas,
- X is Σ_1 on A iff it is Π^1_1,
- X is A-finite iff it is hyperarithmetical.

- We write $\mathcal{M} \preceq_\infty \mathcal{N}$ if satisfaction of computable infinitary formulas by tuples from $\mathcal{M}$ is the same in $\mathcal{M}$ and $\mathcal{N}$.
- For a computable ordinal α, we write $\mathcal{M} \preceq_\alpha \mathcal{N}$ if satisfaction of computable Σ_α formulas by tuples from $\mathcal{M}$ is the same in $\mathcal{M}$ and $\mathcal{N}$.

Barwise Compactness

Barwise Compactness. Let A be a countable admissible set and let T be a set of L_A sentences that is Σ_1 on A. If every A-finite subset of T has a model, then T has a model.

Barwise-Kreisel Compactness. Suppose T is a Π^1_1 set of computable infinitary sentences. If every hyperarithmetical subset of T has a model, then T has a model.

Gregory's Theorem

Theorem (Gregory). Let A be a countable admissible set. Suppose T is a set of L_A-sentences that is Σ_1 on A. If T has a pair of countable models $\mathcal{M}, \mathcal{N}$ s.t. $\mathcal{M} \prec_{L_A} \mathcal{N}$, then T has an uncountable model.

Gregory's Theorem is a variant of the "two-cardinal theorem" of Vaught that figured in Morley's Categoricity Theorem. The proof is quite different. Vaught produced an uncountable model as the union of a chain of models, all isomorphic. In the setting of Gregory's Theorem, the theory T may guarantee that all elements satisfy different types. We will see an example later.

Comments on Gregory's Theorem

Gregory's proof was quite clever. There is a simpler proof, using Ressayre's notion of Σ-saturation.

Gregory said that there were known examples showing that the assumption T is Σ_1 on A cannot be dropped. He did not give an example, and we have been unable to find one published.

Recent work in the spirit of Barwise

There is ongoing work on absoluteness of statements asserting the existence of an uncountable member of a "non-elementary" class K—K may be the class of models of a sentence of $L_{\omega_1\omega}$ or $L_{\omega_1\omega}(Q)$, or it may be an "abstract elementary class".

John Baldwin is involved in joint work of this kind with Martin Körwein, Typani Hyttinen, and Sy Friedman, and also with Paul Larson.

Baldwin asked for an example illustrating Gregory's Theorem. He believed (correctly) that the example would involve computability.

Example

Johnson-K-Ocasio-VanDenDriessche. There is a set T of computable infinitary sentences, in a computable language L, s.t. T has just two models, $\mathcal{M}$ and $\mathcal{N}$, up to isomorphism, where $\mathcal{M}, \mathcal{N}$ are countable and $\mathcal{M} \prec_\infty \mathcal{N}$. Moreover, for each computable ordinal α, the set of computable Σ_α sentences in T is hyperarithmetical.

What do we use?

- There is no priority construction.
- We use the hyperarithmetical hierarchy.
- We use iterated forcing.
- For various computable ordinals α, we construct families of α-generic sets, taking them to be $\Delta^0_{\alpha+1}$.

α-generic sets and families

Let α be a computable ordinal.

- A set X is *α-generic* if for each Σ^0_α set D of finite partial functions $p : \omega \to 2$, there is some $p \subseteq \chi_X$ s.t. either $p \in D$ or else there is no $q \in D$ with $q \supseteq p$.
- Let $(X_a)_{a \in A}$ be an indexed family of sets—identified with the relation $R = \{(a, n) : a \in A \ \& \ n \in X_a\}$. The family is *$\alpha$-generic* if for each Σ^0_α set D of finite partial functions from $p : A \times \omega \to 2$, there is some $p \subseteq \chi_R$ s.t. either $p \in D$ or else there is no $q \in D$ with $q \supseteq p$.

Fact: For any computable set A, there is an α-generic family R, indexed by A, s.t. R is $\Delta^0_{\alpha+1}$.

Language of T and universes of $\mathcal{M}$ and $\mathcal{N}$

The language of T consists of unary predicates U_n for $n \in \omega$. Each L-structure represents a family of sets. The set represented by an element x is the set of n s.t. $U_n x$ holds.

The universe of $\mathcal{M}$ is an infinite computable set of constants C, partitioned effectively into infinitely many infinite sets C_n. The extra element of $\mathcal{N}$ is a further constant a.

We identify the constants with the sets they represent, once we have determined these sets.

Sets represented in $\mathcal{M}$ and $\mathcal{N}$

The set a will be "hyperarithmetically" generic; i.e., it is α-generic for all computable ordinals α.

We choose an increasing sequence of computable ordinals $(\alpha_n)_{n\in\omega}$ with limit ω_1^{CK}. We suppose that $\alpha_n + \alpha_{n+1} = \alpha_{n+1}$.

For all $c \in C_n$ and all $k < n$, $U_k c$ iff $U_k a$. Apart from this, the elements of C_n will be mutually α_n-generic, and uniformly $\Delta^0_{\alpha_n+1}$.

We choose the set a in advance. We then use iterated forcing to choose the families of sets C_n, first C_0, then C_1, etc.

Sentences of T

We need to show the following.

Proposition.

A. $\mathcal{M}$ and $\mathcal{N}$ are the only models of T, up to isomorphism.

B. $\mathcal{M} \prec_\infty \mathcal{N}$

For **A**, it is enough to note that the computable infinitary theory of $\mathcal{M}$ and $\mathcal{N}$ includes the following.

1. sentences saying that all elements that are not $\Delta^0_{\alpha_n+1}$ satisfy the same predicates U_k for $k \leq n$,
2. sentences saying exactly which $\Delta^0_{\alpha_n+1}$ sets are represented,
3. a sentence saying that distinct elements differ on some U_k.

Truth in $\mathcal{M}$ and $\mathcal{N}$

The control that we have of truth in $\mathcal{M}$ and $\mathcal{N}$ comes from forcing. We do not decide truth in $\mathcal{M}$ or $\mathcal{N}$ directly. Let $\mathcal{M}_n$ be the structure with universe $\cup_{k\leq n} C_k$, and let $\mathcal{N}_n$ be the structure with universe $\cup_{k\leq n} C_k \cup \{a\}$.

We choose a in advance, deciding truth in $\mathcal{N}_n$, for all possible choices of $\mathcal{M}_n$.

When we choose C_n, having already chosen $C_{<n}$, we decide truth in $\mathcal{M}_n$.

Proposition B

Lemma 1. $\mathcal{M}_n \prec_{\alpha_n} \mathcal{M}_{n+1}$. (This implies that $\mathcal{M}_n \prec_{\alpha_n} \mathcal{M}$.)

Lemma 2. $\mathcal{M}_n \prec_{\alpha_n} \mathcal{N}_n$

Lemma 3. $\mathcal{N}_n \prec_{\alpha_n} \mathcal{N}_{n+1}$. (This implies that $\mathcal{N}_n \prec_{\alpha_n} \mathcal{N}$.)

Assuming these lemmas, we finish as follows.

Proposition B. $\mathcal{M} \prec_\infty \mathcal{N}$

Proof. Suppose $\mathcal{N} \models \varphi(\overline{c})$, where $\varphi(\overline{c})$ is computable Σ_{α_n} and $\overline{c}$ is in $\mathcal{M}_n$. Since $\mathcal{M}_n \prec_{\alpha_n} \mathcal{N}_n \prec_{\alpha_n} \mathcal{N}$, $\varphi(\overline{c})$ holds in $\mathcal{N}_n$ and in $\mathcal{M}_n$. Since $\mathcal{M}_n \prec_{\alpha_n} \mathcal{M}$, it also holds in $\mathcal{M}$.

Outline for proof of Lemma 1: $\mathcal{M}_n \prec_{\alpha_n} \mathcal{M}_{n+1}$

- Suppose $\varphi(\overline{u}, x)$ is a computable infinitary Σ_{α_n} formula, $\overline{c} \in \mathcal{M}_n$, $b \in \mathcal{M}_{n+1}$, and $\mathcal{M}_{n+1} \models \varphi(\overline{c}, b)$. Our forcing language is propositional. Let φ be the natural propositional formula saying that $\mathcal{M}_{n+1} \models \varphi(\overline{c}, b)$. Since φ is true, it is forced by some p saying finitely much about finitely many elements of C_{n+1}.

- There is a computable Σ_{α_n} formula $\psi_{\varphi,p}$ characterizing the possible structures $\mathcal{M}_n$ s.t. when we choose C_{n+1}, p will force φ. Since $\psi_{\varphi,p}$ is true in $\mathcal{M}_n$, it is forced by some q saying finitely much about finitely many elements of C_n.

More on Lemma 1

- Let f be a computable permutation of $C_{\leq n+1}$, fixing $C_{<n}$, $\overline{c}$, constants in q, and those in p apart from b, taking b to some $b' \in C_n$ s.t. $f(p)$ is true, mapping C_n to $C_n - \{b'\}$ and $C_{n+1} - \{b\}$ to C_{n+1}.

- Let $\mathcal{M}'_n$ be the substructure of $\mathcal{M}_{n+1}$ with universe $f(C_{\leq n}) - \{b'\}$. Since $f(q) = q$ is true in $\mathcal{M}'_n$, $f(\psi_{\varphi,p})$ is true in $\mathcal{M}'_n$. Now, $f(\psi_{\varphi,p})$ says that when we extend $\mathcal{M}'$ to a structure with universe $C_{\leq n+1}$, $f(p) \Vdash f(\varphi)$. When we extend $\mathcal{M}'_n$ to $\mathcal{M}_{n+1}$, $f(p)$ forces $f(\varphi)$. Now, $f(\varphi)$ says that $\mathcal{M}_{n+1} \models \varphi(\overline{c}, b')$.

Lemmas 2 and 3 are proved in a similar way. This is all I will say about the example.

I will state some more open questions, not related to Gregory's Theorem, but representing another possible connection between computability and model theory.

Background on Scott rank

The *Scott rank* of a countable structure is a measure of model theoretic complexity. There is more than one definition in use. Instead of choosing one, let us consider the possible values for hyperarithmetical structures.

Well-known facts. Suppose $\mathcal{A}$ is hyperarithmetical. Then $SR(\mathcal{A}) \leq \omega_1^{CK} + 1$.

1. If there is a computable ordinal α s.t. the orbits of all tuples are defined by computable Σ_α formulas, then $SR(\mathcal{A}) < \omega_1^{CK}$.

2. If the orbits of all tuples are defined by computable infinitary formulas, but there is no bound as in 1, then $SR(\mathcal{A}) = \omega_1^{CK}$.

3. If there is a tuple whose orbit is not defined by any computable infinitary formula, then $SR(\mathcal{A}) = \omega_1^{CK} + 1$.

Examples

1. Computable ordinals have computable Scott rank.

2. The Harrison ordering, of order type $\omega_1^{CK}(1+\eta)$, has Scott rank $\omega_1^{CK}+1$. So does the set of c.e. subsets of ω, with $\subseteq$.

3. There are computable structures of Scott rank ω_1^{CK}—a tree, a field, a linear ordering, a 2-step nilpotent group, etc. For the examples we really know, the computable infinitary theory is $\aleph_0$-categorical.

Note: ω_1^{CK} has Scott rank ω_1^{CK}. The computable infinitary theory of ω_1^{CK} is the same as for $\omega_1^{CK}(1+\eta)$. Any sentence in the theory is true in some computable ordinal.

Questions of Goncharov-K

1. Is there a computable structure $\mathcal{A}$ of high Scott rank s.t. $\mathcal{A}$ is the only computable model (up to isomorphism) of some computable infinitary sentence?

2. Suppose $\mathcal{A}$ is a computable structure of high Scott rank. Is it true that for all Σ^1_1 sets S, there a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ s.t. $n \in S$ iff $\mathcal{C}_n \cong \mathcal{A}$?

These questions are related to Vaught's Conjecture. Antonio Montalbán suggested (on Wednesday) that if Vaught's Conjecture fails, then a slight variant of Question 1 should have a positive answer. The structure $\mathcal{A}$ lives in a countable admissible set A, $\omega_1^{\mathcal{A}} = \alpha$ is the set of ordinals in A, the sentence φ is in L_A, and $\mathcal{A}$ is the unique model of φ, having Scott rank $\geq \alpha$, and living in A (or a fattening).

J. Millar-Sacks. There is a structure $\mathcal{A}$, not hyperarithmetical, but living in a fattening A of the least admissible set, s.t. $SR(\mathcal{A}) = \omega_1^{CK}$, and the L_A theory of $\mathcal{A}$ is not $\aleph_0$-categorical.

Question. Is there a computable structure $\mathcal{A}$ s.t. $SR(\mathcal{A}) = \omega_1^{CK}$ and the computable infinitary theory is not $\aleph_0$ categorical?