Lattice embeddings into the computably enumerable ibT- and cl-degrees

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Lattice embeddings (Definition)

Definition: A *lattice* is a partial order $\mathcal{L} = (L, \leq)$ such that each two elements a, $b \in L$ have a greatest lower bound $a \wedge b$ and a least upper bound $a \vee b$ in L.

Definition: A *lattice embedding* of a lattice $\mathcal{L}$ into a partial order $\mathcal{P} = (P, \leq)$ is a one-to-one function $f : L \to P$ such that

- $a \leq_{\mathcal{L}} b$ iff $f(a) \leq_{\mathcal{P}} f(b)$
- $f(a \vee_{\mathcal{L}} b) = f(a) \vee_{\mathcal{P}} f(b)$
- $f(a \wedge_{\mathcal{L}} b) = f(a) \wedge_{\mathcal{P}} f(b)$

for all $a, b \in \mathcal{L}$.

Examples of embeddings in the Turing degrees I

Theorem (Lachlan 1966 / Yates 1966)

There is a minimal pair $(\mathbf{a}, \mathbf{b})$ *of c.e. Turing degrees. (In particular, there is a pair of incomparable c.e. Turing degrees with a greatest lower bound.)*

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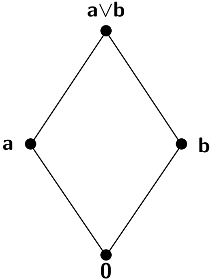
Since the c.e. Turing degrees are an upper semi-lattice, this implies:

Corollary

The diamond is embeddable into $\mathcal{R}_{\mathrm{T}}$ *(preserving the least element).*



Examples of embeddings in the Turing degrees II

Theorem (Lachlan–Thomason / Lerman, 1971)

The countable atomless Boolean algebra can be embedded into $\mathcal{R}_{\mathrm{T}}$.

Corollary

Every finite distributive lattice is embeddable into $\mathcal{R}_{\mathrm{T}}$.

How about nondistributive lattices?

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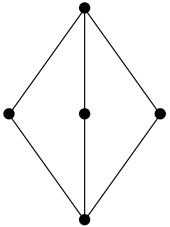
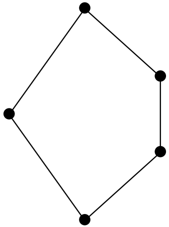
Every finite distributive lattice is embeddable into $\mathcal{R}_{\mathrm{T}}$.

How about nondistributive lattices?

Theorem (Lachlan 1972)

The nondistributive lattices M_3 *and* N_5 *are embeddable into* $\mathcal{R}_{\mathrm{T}}$.





Counterexamples of embeddings in the Turing degrees I

The embeddability of M_5 and N_5 is interesting since every nondistributive lattice contains at least one of them as a sublattice. On the other hand, not all finite nondistributive lattices are embeddable into $\mathcal{R}_{\mathrm{T}}$.

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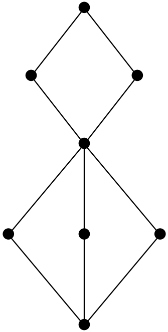
Theorem (Lachlan and Soare, 1980)

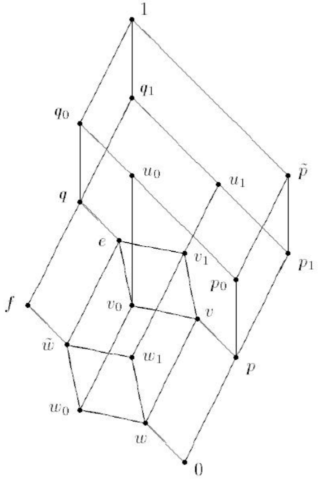
The lattice S_8 cannot be embedded into $\mathcal{R}_{\mathrm{T}}$.



Counterexamples of embeddings in the Turing degrees II

Theorem (Lempp and Lerman, 1997)

The following 20-element lattice cannot be embedded into $\mathcal{R}_{\mathrm{T}}$.



Lattice L_{20}

ibT- and cl-reducibility

Definition (Soare / Downey, Hirschfeldt and LaForte)

A set $A \subseteq \mathbb{N}$ is **computably Lipschitz-(cl)-reducible** to a set $B \subseteq \mathbb{N}$ if A is Turing-reducible to B via a reduction Φ such that for the use function φ of this reduction,

$$(\forall x)\varphi(x) \leq x + c$$

for some constant c. We write $A \leq_{\mathrm{cl}} B$. If c can be chosen to be 0, we say that A is **identity-bounded Turing-(ibT-)reducible** to B (written $A \leq_{\mathrm{ibT}} B$).

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Note that this is a very strong restriction of bounded Turing- (aka weak truth-table-)reducibility, where the size of oracle questions is bounded by some computable function. Here, this function is required to be the identity function [plus a constant].

Definition: For $r = \mathrm{cl}, \mathrm{ibT}$, an r-degree is called c.e. if it contains a c.e. set. $\mathcal{R}_r$ is the partial ordering of the c.e. r-degrees.

Fundamental facts about $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$

Let $r \in \{\mathrm{ibt}, \mathrm{cl}\}$.

- Barmpalias 2005: There are no maximal elements in $\mathcal{R}_r$. (Proof for $r = \mathrm{ibt}$: A noncomputable $\Rightarrow A <_{\mathrm{ibt}} A - 1 = \{x - 1 : x \in A \text{ and } x > 0\}$.) In particular, there are no complete c.e. r-degrees.

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- Barmpalias / Fan and Lu 2005: There are maximal pairs (= pairs without common upper bound) of c.e. r-degrees. In particular, $\mathcal{R}_r$ is not an upper semi-lattice.
- ibT-cl-Conversion Lemma (Ambos-Spies, Ding, Fan, Merkle): Let $A, B_0, \ldots, B_n (n \geq 0)$ be c.e. sets such that

  $$deg_{\mathrm{ibT}}(B_0) \vee \ldots \vee deg_{\mathrm{ibT}}(B_n) = deg_{\mathrm{ibT}}(A).$$

  Then

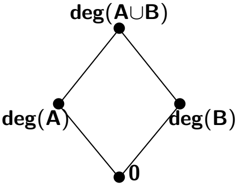
  $$deg_{\mathrm{cl}}(B_0) \vee \ldots \vee deg_{\mathrm{cl}}(B_n) = deg_{\mathrm{cl}}(A).$$

  The same holds for $\wedge$ instead of $\vee$.

Embedding distributive lattices

Theorem

For $r = \mathrm{ibT}, \mathrm{cl}$, the diamond is embeddable into $\mathcal{R}_{\mathrm{r}}$ (preserving the least element).



Theorem (Ambos-Spies)

For $r = \mathrm{ibT}, \mathrm{cl}$, the countable atomless Boolean algebra can be embedded into $\mathcal{R}_{\mathrm{r}}$.

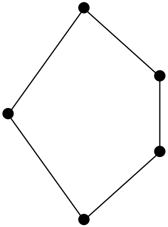
Corollary

Every finite distributive lattice is embeddable into $\mathcal{R}_{\mathrm{r}}$.

Embedding nondistributive lattices into the c.e. ibT- and cl-degrees

Theorem (Ambos-Spies, Bodewig, Kräling, and Yu)

For $r = \mathrm{ibT}, \mathrm{cl}$, the nondistributive lattice N_5 is embeddable into $\mathcal{R}_r$ (preserving the least element).



Embedding the N_5

Requirements and how to satisfy them:

- $A \leq_{\mathrm{ibT}} B \leq_{\mathrm{ibT}} D$ and $C \leq_{\mathrm{ibT}} D$
  by permitting.
- $B \not\leq_{\mathrm{cl}} A$
  We satisfy the requirements $B \neq \Phi^A$ for cl-functionals Φ with $\varphi(x) \leq x + e$ by enumerating numbers x into B and restraining $A \upharpoonright x + e + 1$. (Diagonalization)
- $deg_{\mathrm{ibT}}(B) \wedge deg_{\mathrm{ibT}}(C) = deg_{\mathrm{ibT}}(\emptyset)$
  with the minimal pair technique.
- $deg_{\mathrm{ibT}}(A) \vee deg_{\mathrm{ibT}}(C) = deg_{\mathrm{ibT}}(D)$
  We satisfy the requirements
  $A = \Phi^W$ and $C = \Psi^W \implies D \leq_{\mathrm{ibT}} W$
  for all c.e. sets W and all ibT-functionals Φ, Ψ.

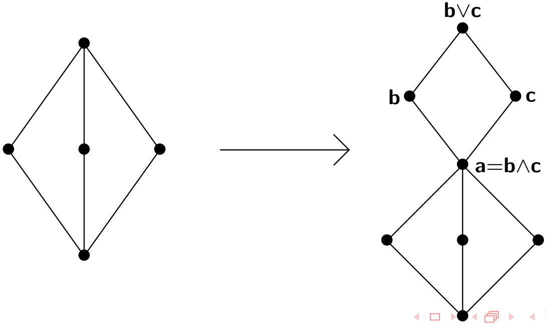
Embedding the S_8 into $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$

Theorem (Ambos-Spies, Bodewig, Kräling, and Yu)

Let $r = \mathrm{ibT}, \mathrm{cl}$. Every c.e. r-degree **a** *is branching, i.e. is the greatest lower bound of incomparable c.e. r-degrees* **b** *and* **c**. *Moreover, it is possible to choose* **b** *and* **c** *such that* $\mathbf{b} \vee \mathbf{c}$ *exists.*

Corollary

If the M_3 is embeddable, then the S_8 is embeddable into $\mathcal{R}_r$.



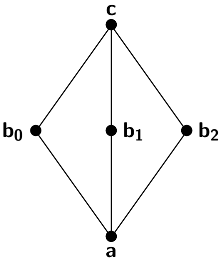
Embedding the M_3 into $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$

Theorem (Ambos-Spies and Wang)

The nondistributive lattice M_3 cannot be embedded into $\mathcal{R}_{\mathrm{ibT}}$ or $\mathcal{R}_{\mathrm{cl}}$ preserving the least element.

Theorem (Ambos-Spies, Bodewig, Kräling, and Wang)

The M_3 is embeddable into $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$.



Embedding the M_3

Requirements and how to satisfy them:

- $A \leq_{\mathrm{ibT}} B_i \leq_{\mathrm{ibT}} C$ for $i \in \{0, 1, 2\}$
  by permitting.
- $B_i \not\leq_{\mathrm{cl}} A$ for $i \in \{0, 1, 2\}$
  We satisfy the requirements $B \neq \Phi^A$ for cl-functionals Φ with $\varphi(x) \leq x + e$ by enumerating numbers x into B and restraining $A \upharpoonright x + e + 1$. (Diagonalization)
- $deg_{\mathrm{ibT}}(B_i) \wedge deg_{\mathrm{ibT}}(B_j) = deg_{\mathrm{ibT}}(A)$ for $i \neq j, i, j \in \{0, 1, 2\}$
  with the minimal pair technique relative to A.
- $deg_{\mathrm{ibT}}(B_i) \vee deg_{\mathrm{ibT}}(B_j) = deg_{\mathrm{ibT}}(C)$ for $i \neq j, i, j \in \{0, 1, 2\}$
  We satisfy the requirements
  $B_i = \Phi^W$ and $B_j = \Psi^W \implies C \leq_{\mathrm{ibT}} W$
  for all c.e. sets W and all ibT-functionals Φ, Ψ.

Making the strategies work together - a naive approach

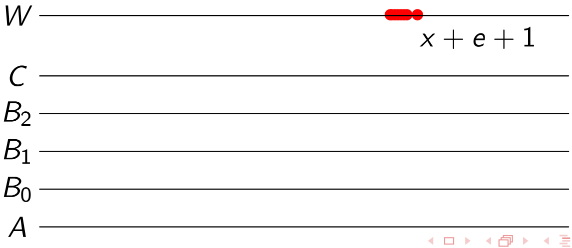
Given the basic strategies to satisfy the requirements, we need them to work together. A naive approach to satisfy the requirement $B_0 \neq \Phi(A)$ would look like this:

1. Wait until $\Phi(A)(x) \downarrow= 0$ for some diagonalisation witness x, and enumerate x into B_0.
2. For $B_0 \leq_{\mathrm{ibT}} C$, enumerate some number $y \leq x$ into C.
3. For the join requirement $deg_{\mathrm{ibT}}(C) = deg_{\mathrm{ibT}}(B_1) \vee deg_{\mathrm{ibT}}(B_2)$, enumerate some number $z \leq y$ into B_1 or B_2, say into B_2.
4. For the meet requirement $deg_{\mathrm{ibT}}(A) = deg_{\mathrm{ibT}}(B_0) \wedge deg_{\mathrm{ibT}}(B_2)$, enumerate some number $w \leq \max(x, z) = x$ into A.

But this conflicts with restraining $A \upharpoonright (x + e + 1)$! Hence the approach fails!

Making the strategies work together - a modified approach

Assume we want to satisfy $B_0 \neq \Phi_e(A)$, and we have a witness x with $[x+1, x+e+1] \subseteq W$ for a certain c.e. set W.



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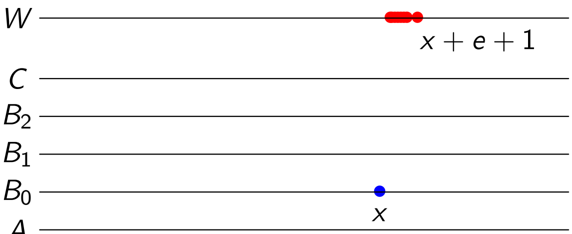

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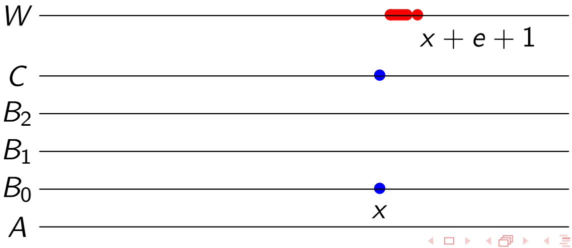
2. Enumerate x into C to make $B_0 \leq_{\mathrm{ibT}} C$.



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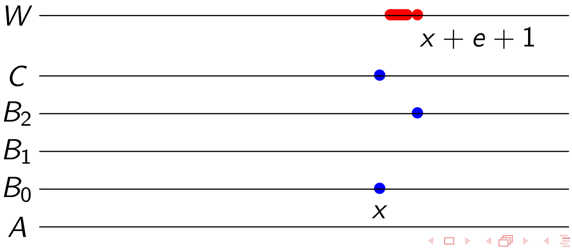
1. Wait until $\Phi(A)(x) \downarrow = 0$. Then enumerate x into B_0.
2. Enumerate x into C to make $B_0 \leq_{\mathrm{ibT}} C$.
3. To make $C \leq_{\mathrm{ibT}} W$, enumerate $x+e+1$ into B_2 .



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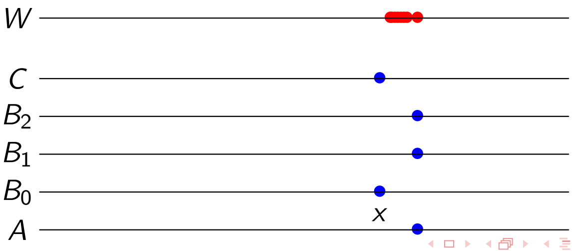
3. To make $C \leq_{\mathrm{ibT}} W$, enumerate $x + e + 1$ into B_2 .
4. To make $deg_{\mathrm{ibT}}(A) = deg_{\mathrm{ibT}}(B_0) \wedge deg_{\mathrm{ibT}}(B_2)$, enumerate $x + e + 1$ into A and B_1 at the same time.



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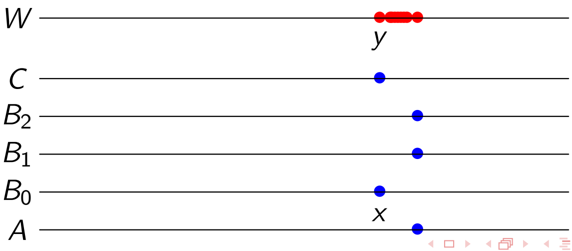
4. To make $deg_{\mathrm{ibT}}(A) = deg_{\mathrm{ibT}}(B_0) \wedge deg_{\mathrm{ibT}}(B_2)$, enumerate $x+e+1$ into A and B_1 at the same time.
5. W has to react by the enumeration of some number $y \leq x+e+1$, hence $y \leq x$.



Safe positions

To make this work, the diagonalisation requirements (for Φ_e) have to choose their witnesses x in such a way that the interval $[x+1, x+e+1]$ is contained in W when the diagonalisation starts.

Definition

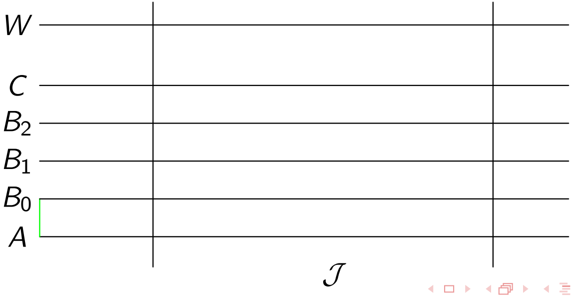
An interval $\mathcal{I}$ is **safe for** W at stage s if $\mathcal{I} \subseteq W$ and $\mathcal{I} \cap (A \cup B_0 \cup B_1 \cup B_2) = \emptyset$ at stage s.

How can we create safe intervals?

Creating safe intervals for one W

If $\Phi(W) = B_1$ and $\Psi(W) = B_2$, we can create a safe interval as follows. At the beginning, we reserve an interval $\mathcal{J}$ that contains no elements from A, B_0, B_1, B_2 or C. Enumerate the elements from $\mathcal{J}$ from right to left, each element first into B_1, then into B_2 and wait for W to respond by a smaller or equal enumeration. Also care for permitting by C.

Due to lack of space, if $\mathcal{J}$ is long enough, we get a safe subinterval $\mathcal{I}$ of the required length.



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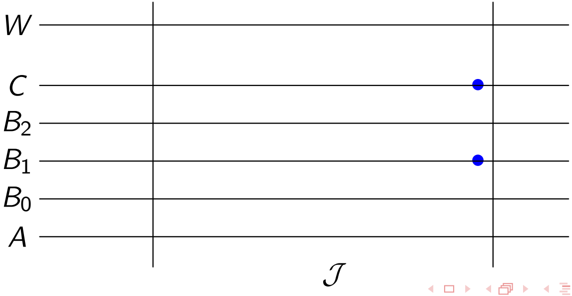

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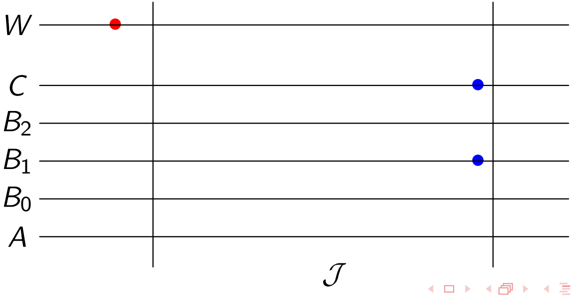

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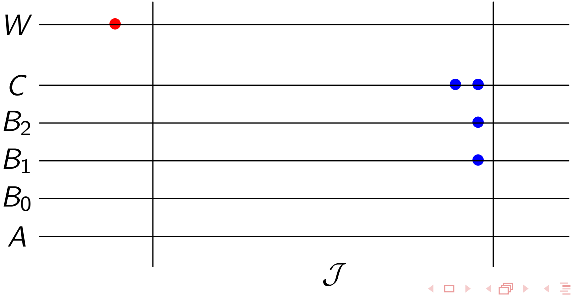

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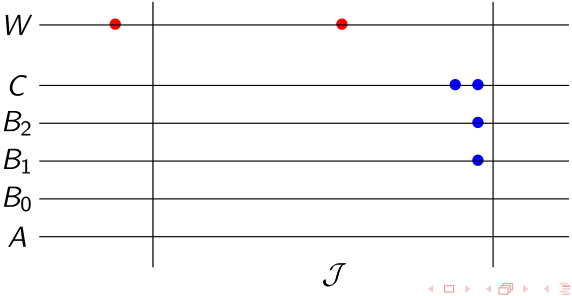

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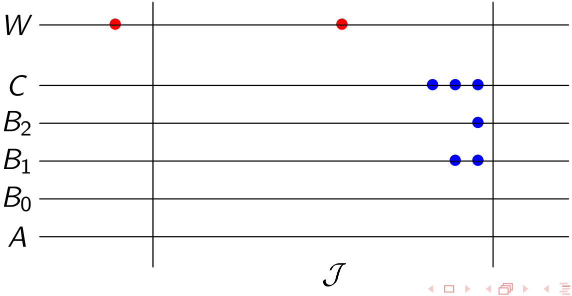

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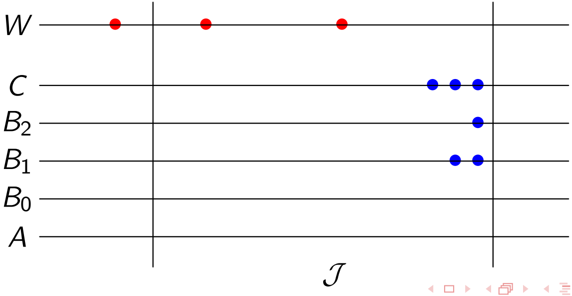

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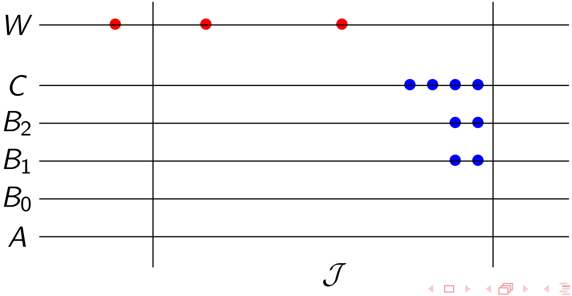

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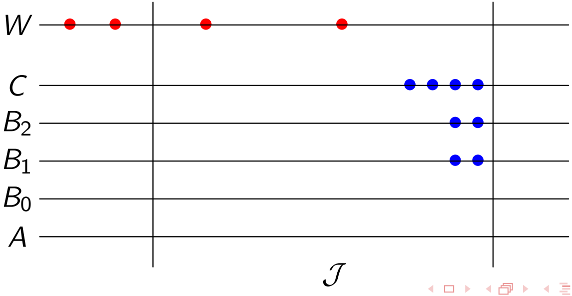

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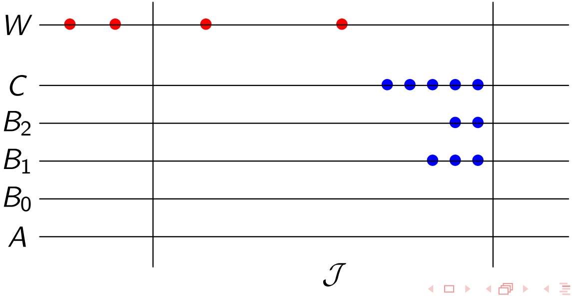

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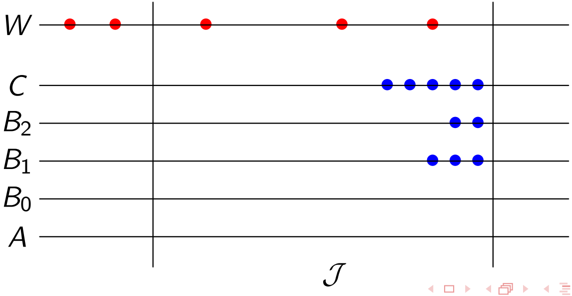

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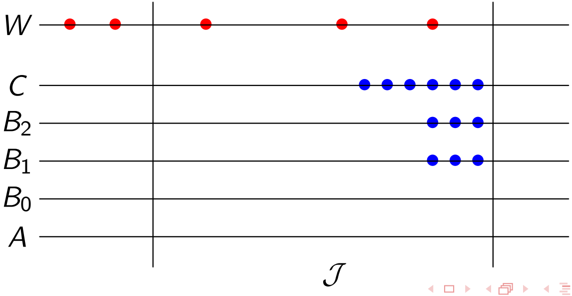

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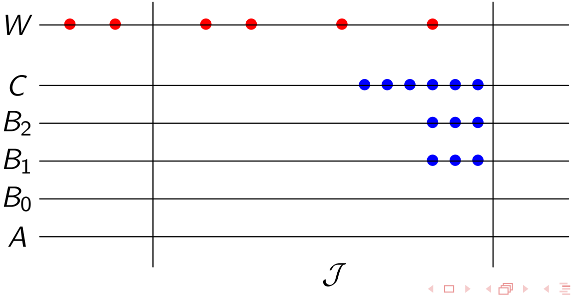

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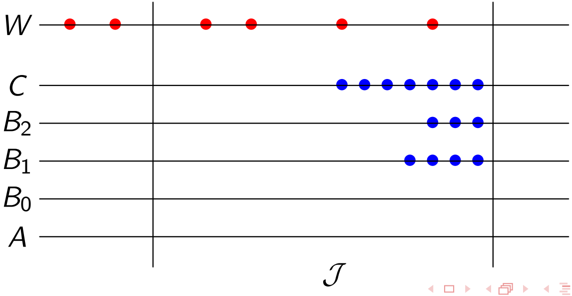

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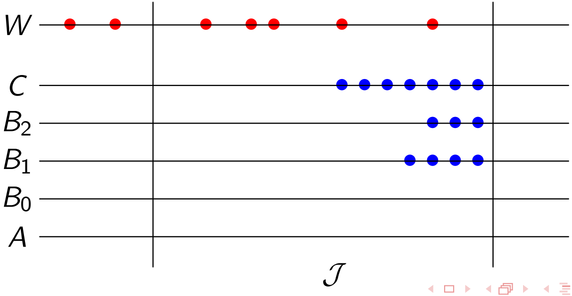

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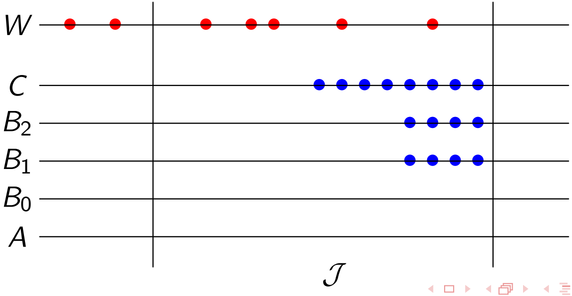

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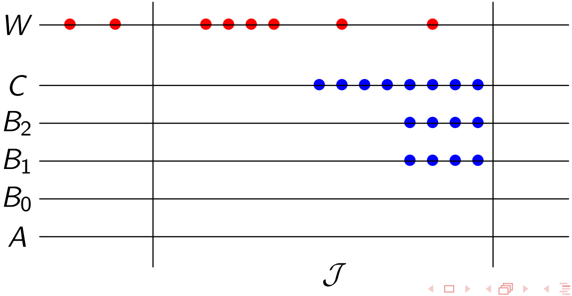

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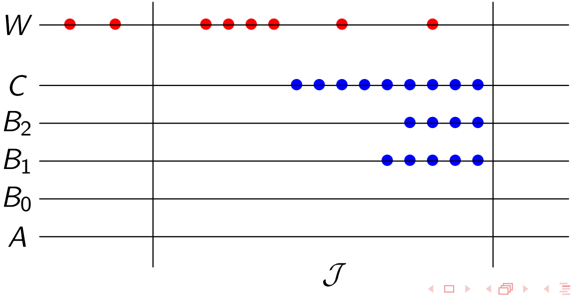

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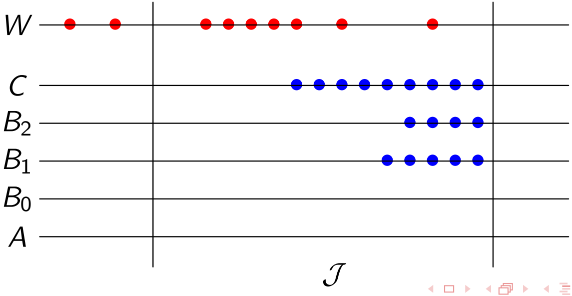

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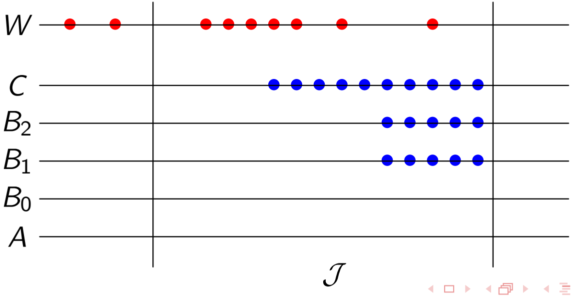

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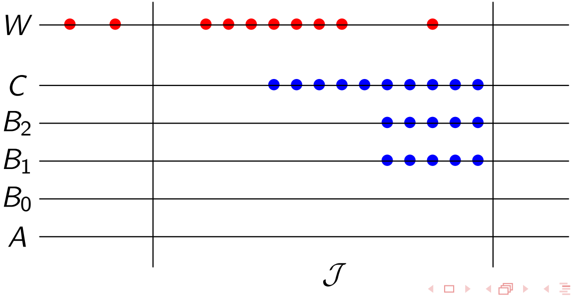

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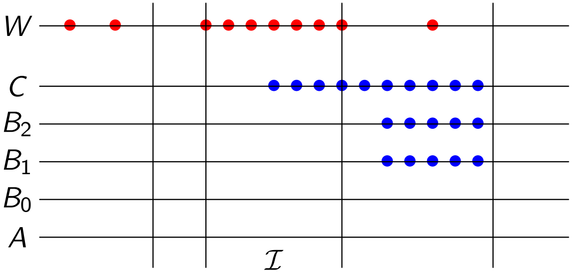

Creating safe intervals for two W

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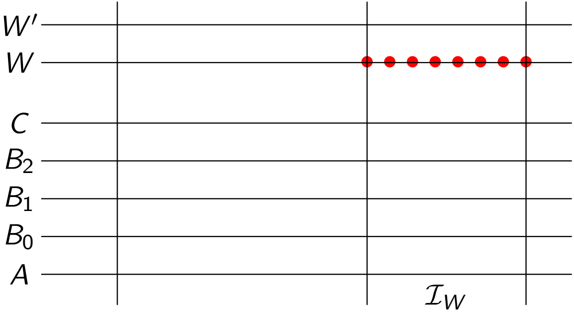

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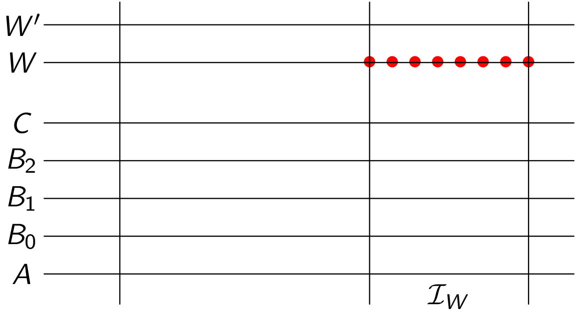

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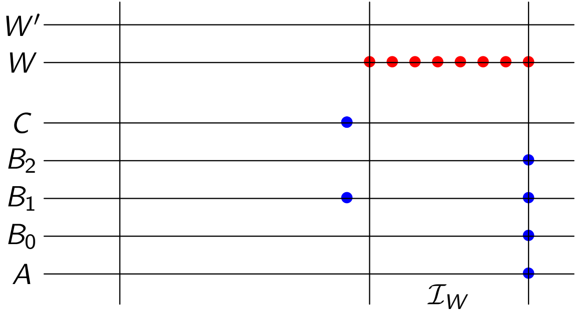

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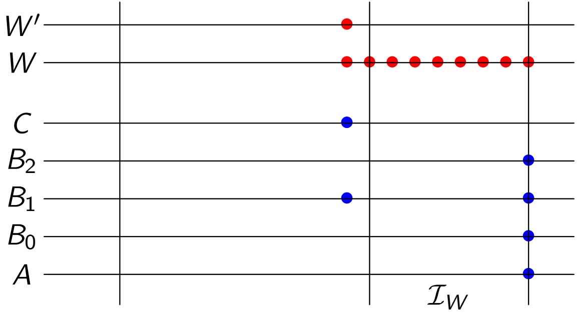

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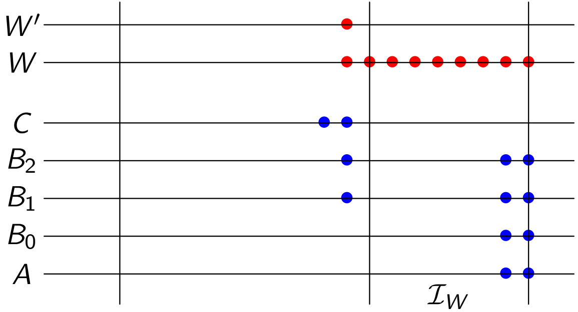

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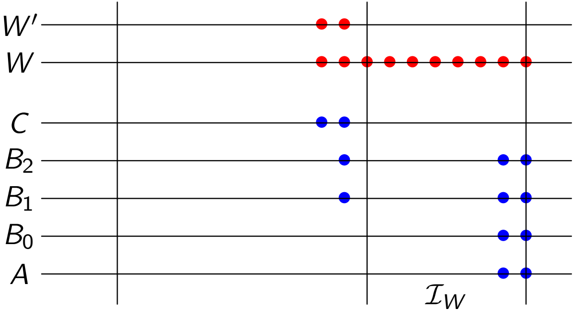

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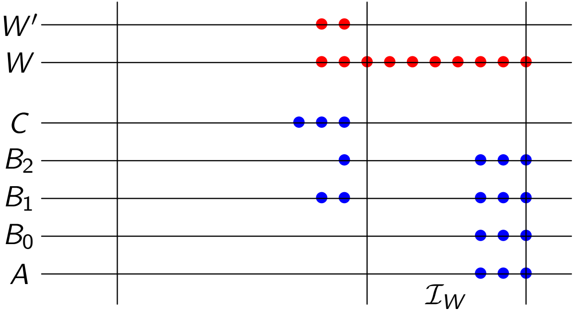

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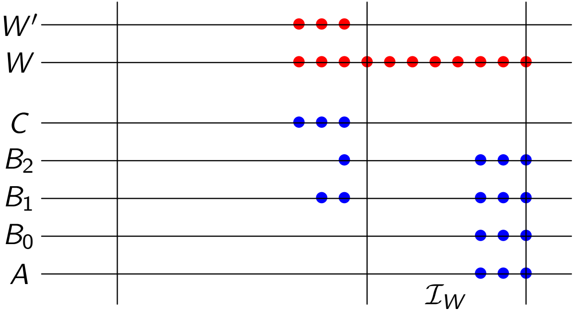

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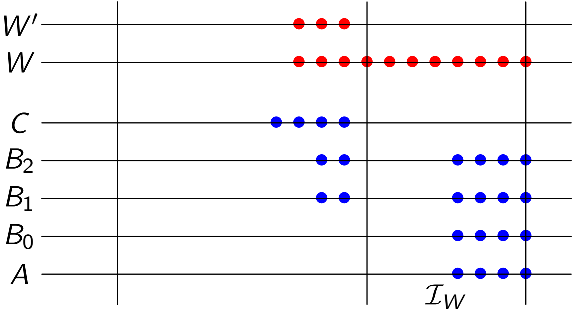

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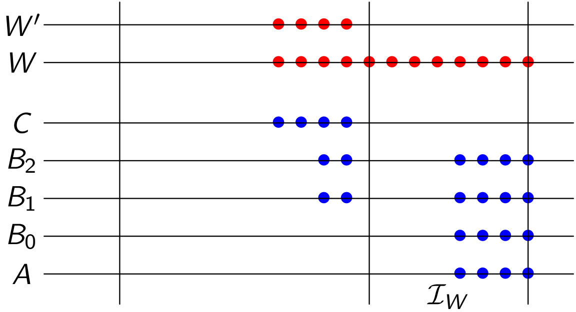

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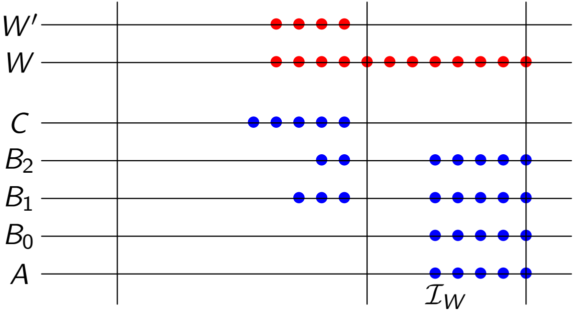

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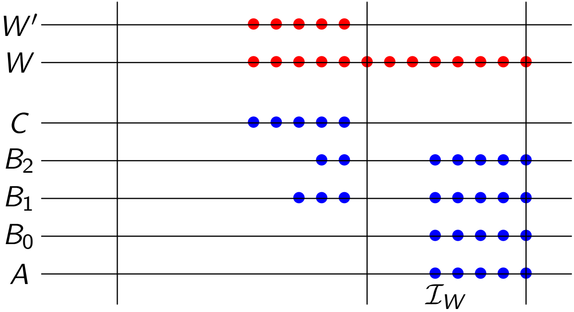

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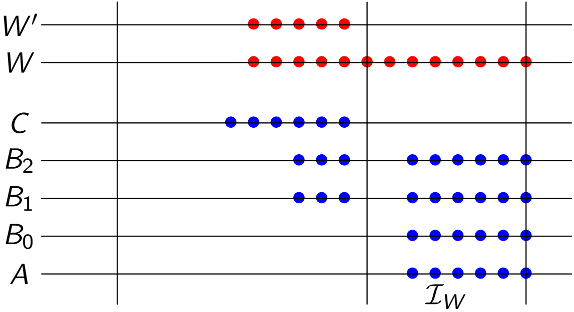

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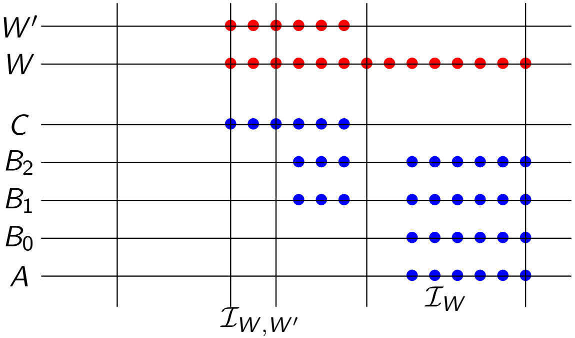
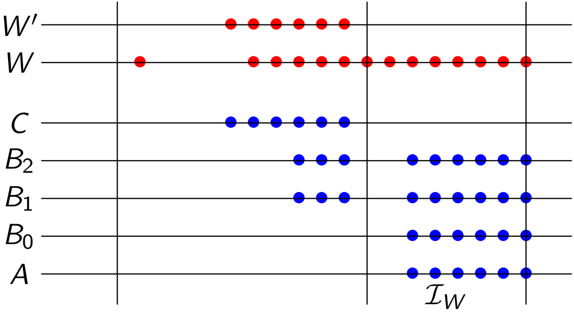


It is possible that the responses of W contain gaps. Then we don't get a safe interval which is long enough.
In this case we try again to create a new $\mathcal{I}_W$ below the last enumeration into C. W can never use elements from the gap to respond to our enumerations.
If we do this enough times, W has no space left to produce gaps.
But we need to start with a VERY long interval $\mathcal{J}$!



It is possible that the responses of W contain gaps. Then we don't get a safe interval which is long enough.

In this case we try again to create a new $\mathcal{I}_W$ below the last enumeration into C. W can never use elements from the gap to respond to our enumerations.

If we do this enough times, W has no space left to produce gaps.

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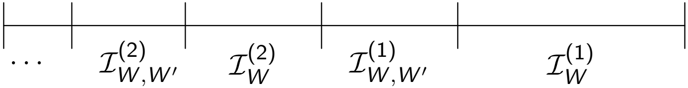

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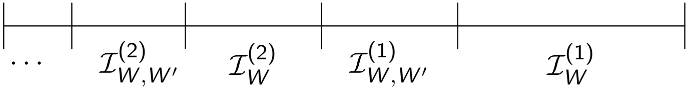

The case which respects finitely many sets $W_0, \dots, W_n$ needs even more space ($|\mathcal{J}| \geq 5,000,000$ for $n = 4$).

Open questions

- Are all finite lattices embeddable into $\mathcal{R}_{\mathrm{ibT}}$ and $\mathcal{R}_{\mathrm{cl}}$ or is there a counterexample?
- Which lattices can be embedded into $\mathcal{R}_{\mathrm{ibT}}$ or $\mathcal{R}_{\mathrm{cl}}$ preserving the least element?

Thank you for your attention!