The Typical Turing Degree

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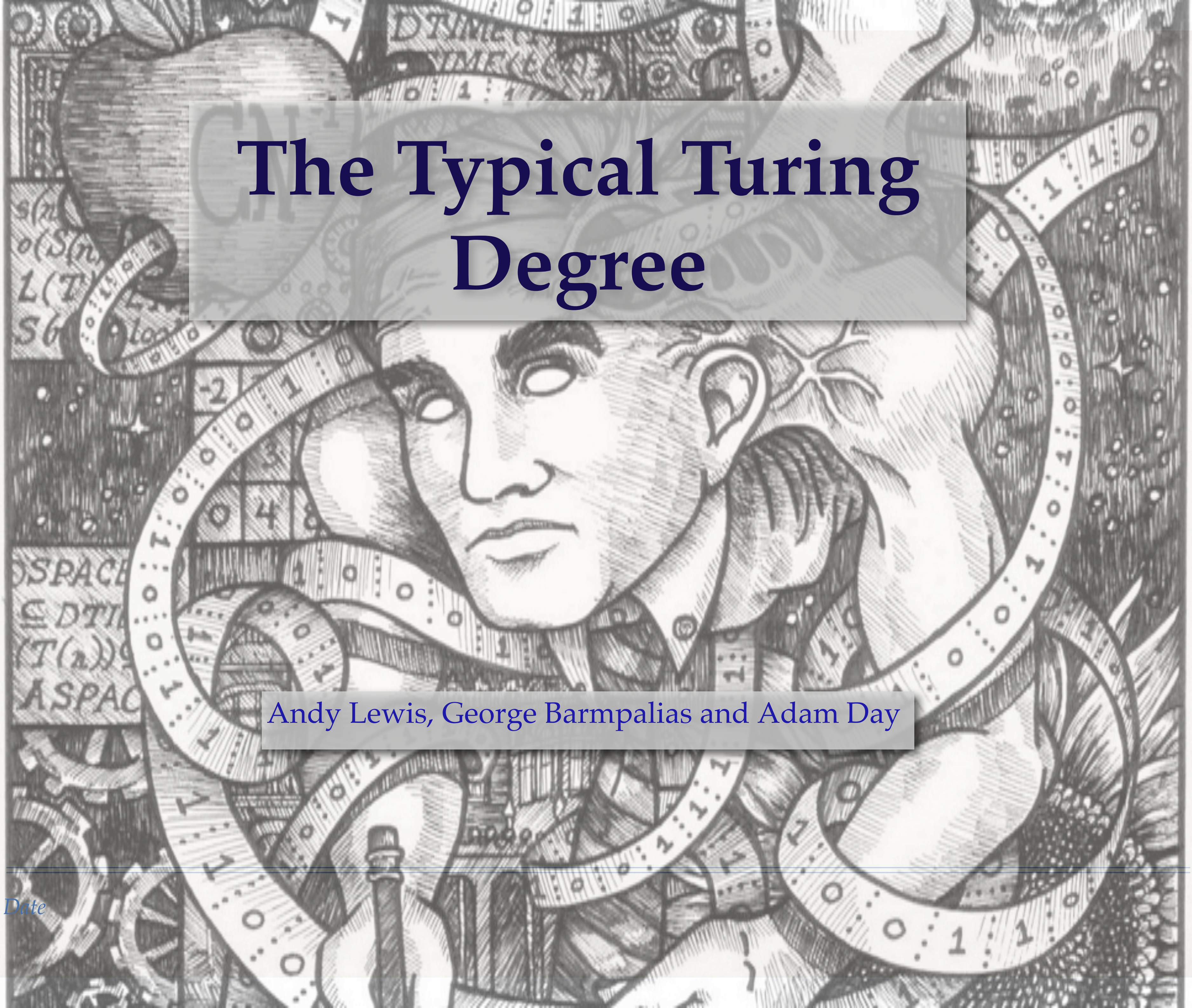

Date

KOLMOGOROV'S 0-1 LAW

Theorem. Every measurable tailset is of measure 0 or 1.

Very simple proof – basically if $[W]$ is an open covering of P, with $\mu([W]) = \alpha$, then let $W^* = \{\sigma * \tau | \ \sigma, \tau \in W\}$. We have $\mu([W^*]) = \alpha^2$, and since P is a tailset $[W^*]$ is still an open covering of P.

MARTIN-LÖF RANDOMNESS

A set $B \subseteq \omega$ is Martin-Löf random if it does not belong to any **effectively null** set. A set $P \subseteq 2^\omega$ is effectively null if there is an algorithm which, given any rational $\epsilon > 0$ as input, enumerates a set of finite binary strings W_ϵ such that $[W_\epsilon]$ is an open covering of P of measure $< \epsilon$.

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So now the crucial observation is just this: if a set is null, then it is effectively null relative to some oracle. **So** if a set is of measure 1, then there is some level of randomness which suffices to ensure that you belong to the set.

MEASURABLE SETS

In the below, 'definable' means definable as a subset of the Turing degrees in the language for the structure (i.e., the language of partial orders).

Observation. Whether or not all definable sets of Turing degrees are measureable is independent of ZFC.

In other words, whether or not we can take any definable property of the Turing degrees, and force a degree to satisfy it, or else force a degree to satisfy its negation, simply by insisting that it be sufficiently random, this is independent of ZFC.

BAIRE CATEGORY

Recall that a set is meager if it is a countable union of nowhere dense sets.

Banch-Mazur games:

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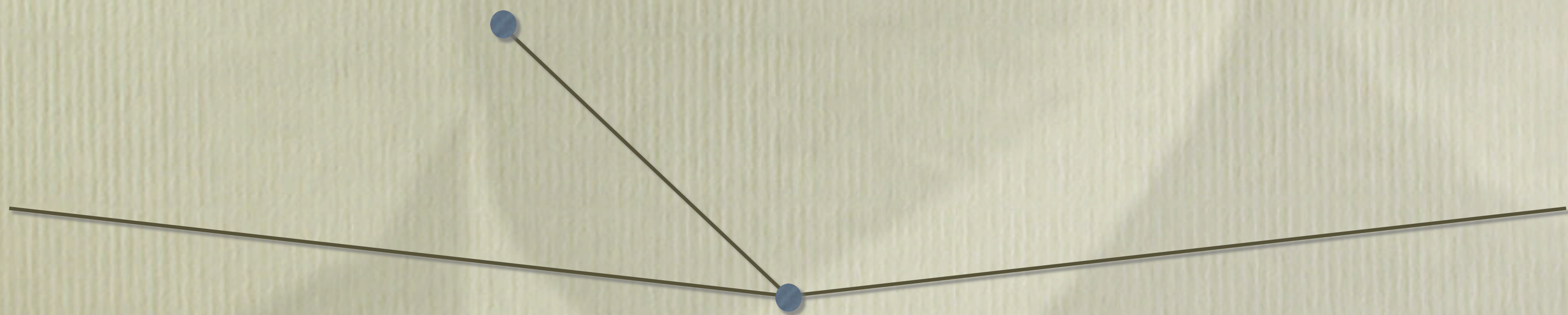

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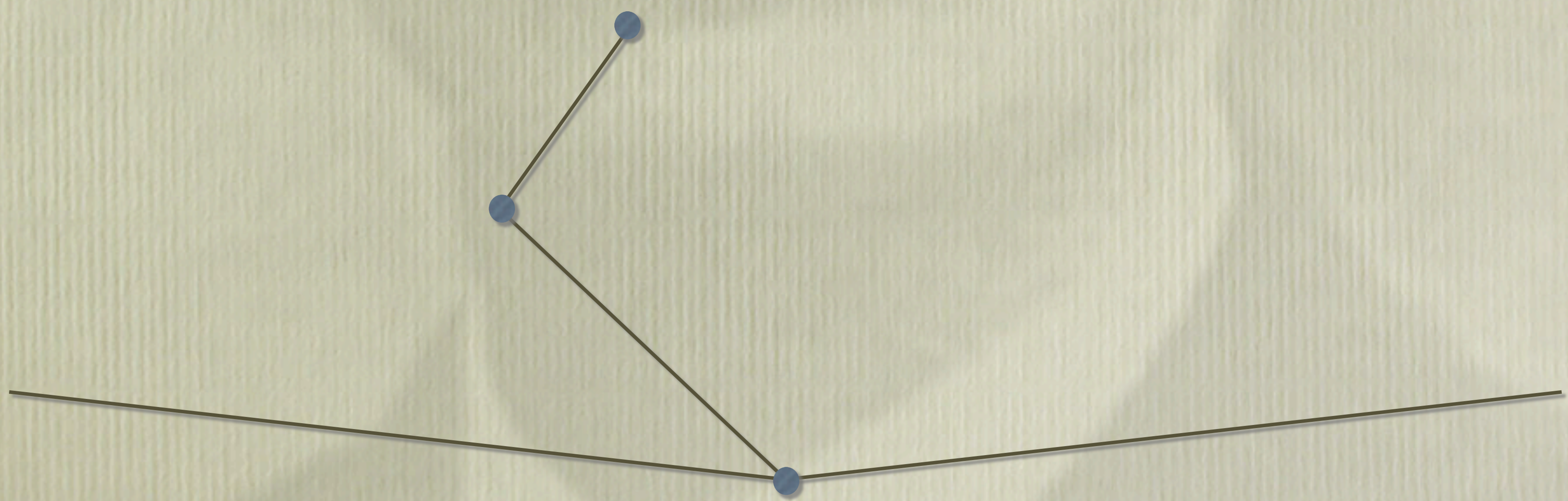

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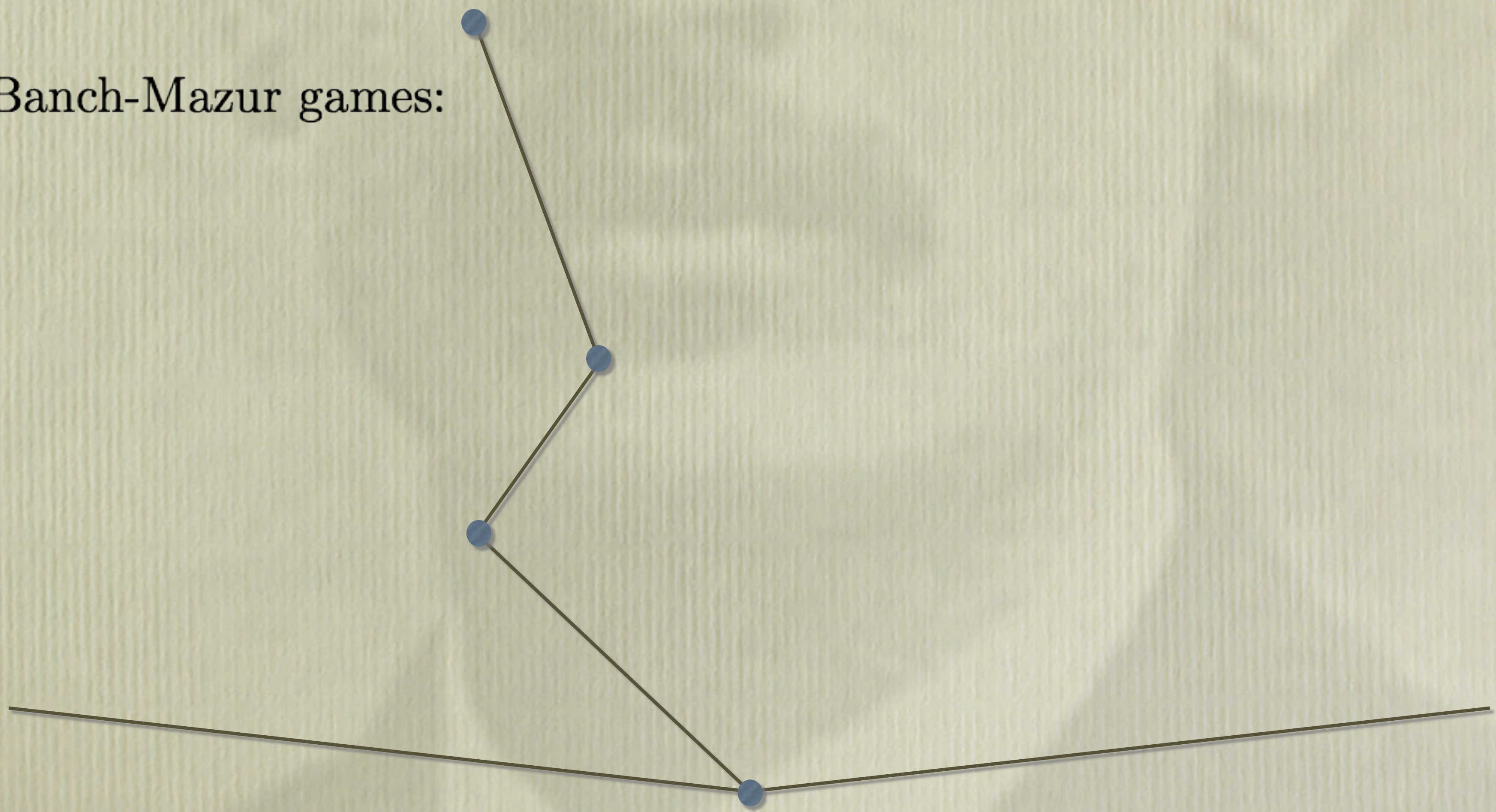


GENERIC SETS

The role that was played by randomness in the context of measure is now played by genericity (where B is 1-generic relative to A if, for every W which is c.e. in A, $\exists \sigma \subset B[(\sigma \in W) \ \vee \ (\forall \tau \supset \sigma)[\tau \notin W]]$).

If a set is comeager then there is some level of genericity which suffices to ensure membership.

MEAGER AND COMEAGER SETS

Observation. Whether or not all definable sets of degrees are either meager or comeager is independent of ZFC.

In other words, whether or not we can take any definable property of the Turing degrees and force a degree to satisfy it, or else force a degree to satisfy its negation, simply by requiring that it be sufficiently generic, this is independent of ZFC.

TO BE CONSIDERED...

Minimality

Bounding a minimal

The join property

The framework

An heuristic principle

The cupping property

Being a minimal cover

Being the top of a diamond

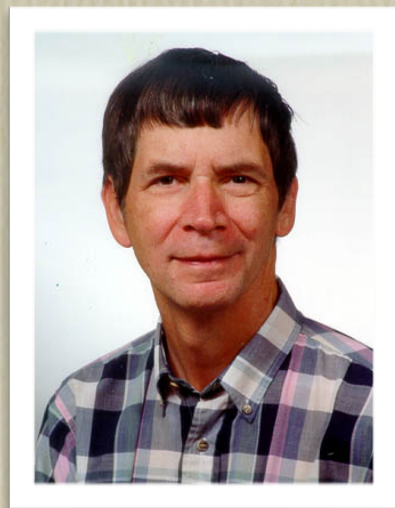
The meet and complementation properties

MINIMAL DEGREES

Theorem [Sacks]. The minimal degrees are of measure 0.

In fact it is easy to see that no 1-random set A is of minimal degree. Writing $A = B \oplus C$ (so B gives the even bits of A and C gives the odd bits) we see that B and C must be Turing incomparable.

Theorem [Jockusch]. The minimal degrees are meager. No 1-generic is of minimal degree.



DEGREES WHICH BOUND MINIMALS.

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Background

Theorem [Jockusch]. Every generalized high degree bounds a minimal degree.

Theorem [Ellison,L]. Every generalized high degree is the join of two minimal degrees.

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Theorem [Paris]. The measure of those degrees which bound a minimal degree is 0.

DEGREES WHICH BOUND MINIMALS.

Theorem [Kurtz]. For almost all degrees $\mathbf{a}$, if $\mathbf{0} < \boldsymbol{b} \leq \boldsymbol{a}$, then $\boldsymbol{b}$ bounds a 1-generic.

Theorem [BDL]. If $\mathbf{a}$ is 2-random and $\mathbf{0} < \boldsymbol{b} \leq \boldsymbol{a}$, then $\boldsymbol{b}$ bounds a 1-generic.

Corollary [BDL]. If $\mathbf{a}$ is 2-random then it does not bound any minimal degrees.

DEGREES WHICH BOUND MINIMALS.

A set is weakly 2-random if it does not belong to any null Π^0_2 class.

A Demuth test is a sequence of c.e. sets $\{V_m\}_{m\in\omega}$ such that $\mu([V_m]) \leq 2^{-m}$ and there is an ω-c.e. function f such that $V_m = W_{f(m)}$ (i.e. V_m is the $f(m)$th c.e. set). A set A fails the test if there exist infinitely many m such that $A \in [V_m]$. A set is Demuth random if it doesn't fail any Demuth test.

DEGREES WHICH BOUND MINIMALS.

There are Demuth randoms which are Δ^0_2. By Kučera's technique of FPF permitting, any FPF Δ^0_2 degree bounds a non-zero c.e. degree. Yates showed that any non-zero c.e. degree bounds a minimal. So there **are** Demuth randoms which bound minimal degrees.

Theorem [BDL]. There are weakly 2-randoms which are generalized high.

Corollary. There are weakly 2-randoms which bound minimal degrees.

DEGREES WHICH BOUND MINIMALS.

Theorem [Martin]. If a meager set of degrees does not contain **0** and is downward closed amongst the non-zero degrees, then its upward closure is still meager.

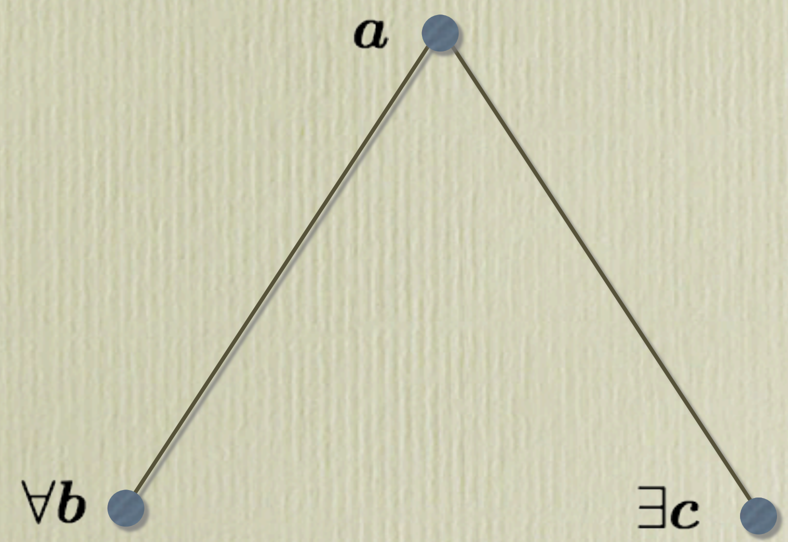
Corollary. The degrees which bound minimals form a meager set.

Theorem [Yates, Jockusch]. 2-generic degrees do not bound minimals.

Theorem [Chong and Downey, also Kumabe]. There are 1-generics which bound a minimal degree.

THE JOIN PROPERTY

We say $\boldsymbol{a}$ satisfies the **join property** if, for all non-zero $\boldsymbol{b} < \boldsymbol{a}$, there exists $\boldsymbol{c} < \boldsymbol{a}$ with $\boldsymbol{b} \vee \boldsymbol{c} = \boldsymbol{a}$.



THE JOIN PROPERTY

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Background

Theorem [Downey, Greenberg, L, Montalbán]. All non-GL_2 degrees satisfy the join property.

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Theorem [B,L]. The measure of the degrees which satisfy the join property is 1. All 2-random degrees satisfy the join property.

Theorem [L]. There are 1-random degrees which do not satisfy the join property.

THE JOIN PROPERTY

The fact that there are 1-randoms which fail to satisfy join follows from the fact [L] that all low FPF degrees fail to satisfy join. Actually, the same reasoning shows that there are Demuth randoms which fail to satisfy join (since there are Demuth randoms which are Δ_2^0 and all Demuth randoms are GL_1).

Question. Are there weakly 2-random degrees which do not satisfy join?

Theorem [BDL]. Every degree below a 2-random satisfies join.

THE JOIN PROPERTY

Theorem [Jockusch]. The degrees which satisfy the join property are comeager. All 2-generics satisfy the join property.

In fact, what Jockusch showed is that all 2-generics satisfy the cupping property. Since Martin showed that if $\mathbf{a}$ is n-generic and $\mathbf{0} < \mathbf{b} \leq \mathbf{a}$ then $\mathbf{b}$ bounds an n-generic, and since the degrees which satisfy the cupping property are upward closed, this means that all non-zero degrees below a 2-generic satisfy cupping.

Theorem [BDL]. All 1-generic degrees satisfy the join property.

AN HEURISTIC PRINCIPLE

Any natural definable property satisfied by all sufficiently random/generic degrees is likely to be satisfied by all non-zero degrees below all sufficiently random/generic degrees.

The reason for believing this is firstly just that it holds for all the properties we have considered so far (as far as the results show). Secondly, according to our frameworks results of the first kind can be translated into results of the second.

THE FRAMEWORK

The strategy for showing that all sufficiently random sets X satisfy a certain degree-theoretic property is as follows:

(a) Translate the property into a countable sequence of requirements $\{R_e\}_{e\in\omega}$ referring to an unspecified set X.

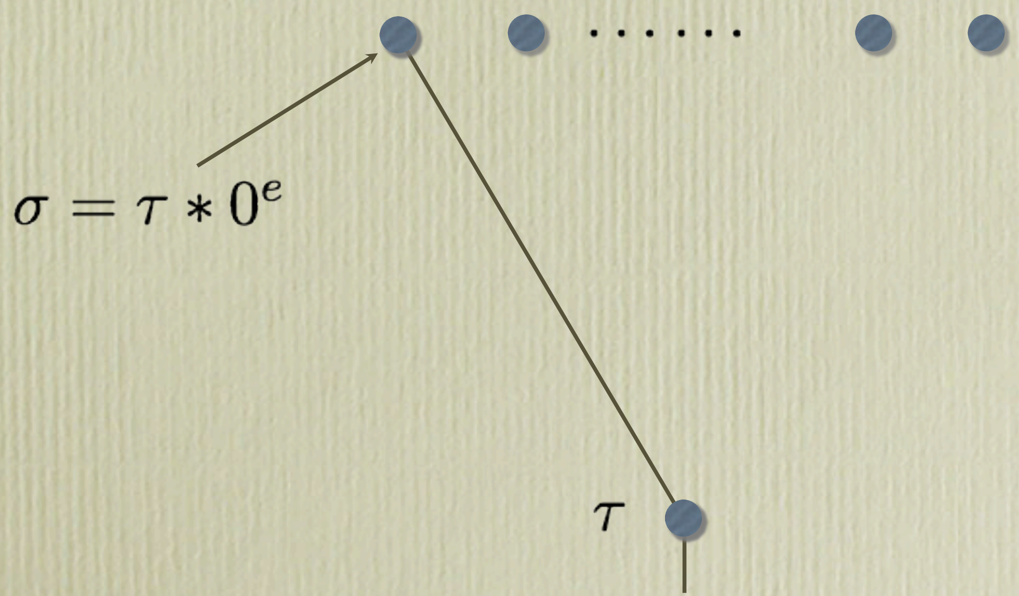
(b) Devise an 'atomic' strategy which takes a number e and a string τ as inputs and satisfies R_e for a certain proportion of extensions X of τ, where this proportion depends on e and not on τ.

(c) Assemble a construction from the atomic strategies in a *standard way*.

EXAMPLE: BOUNDING A 1-GENERIC

(a) $R_e : \exists n\, [\Phi^X \restriction_n \in W_e \ \vee \ \forall \sigma \in W_e,\ \Phi^X \restriction_n \not\subseteq \sigma]$.

(b)



THE FRAMEWORK

The basic rules according to which markers are placed on strings and removed from them are as follows:

(i) At most one marker sits on any string at any given stage.

(ii) If $\tau \subset \tau'$ and at some stage an e-marker sits on τ' and a d-marker sits on τ, then $d \leq e$.

(iii) If a marker is removed from τ at some stage then any marker that sits on any extension of τ is also removed.

THE FRAMEWORK

The outcome of the construction with respect to a particular real X will be reflected by the permanent markers that are placed on initial segments of X. In particular, one of the following outcomes will occur:

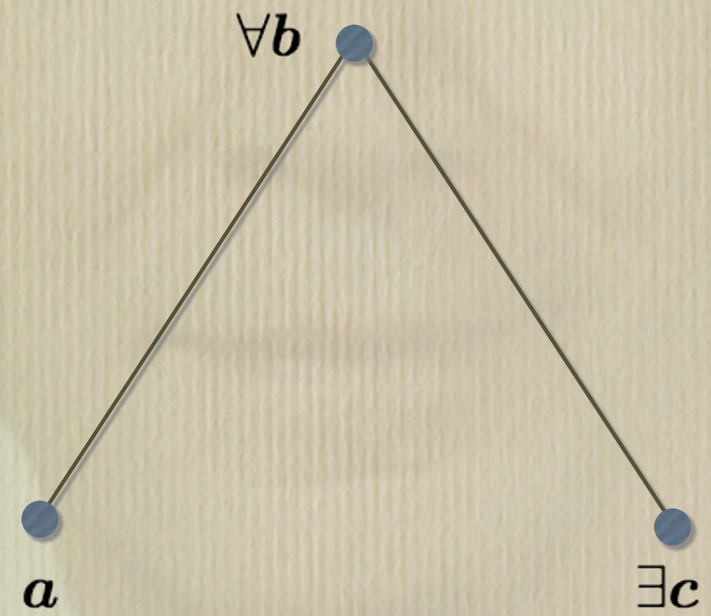
(1) For every $e \in \omega$ there is a permanent e-marker placed on some initial segment of X.

(2) There exists some $e \in \omega$ such that, for each $d \leq e$, a permanent d-marker is placed on an initial segment of X, and such that infinitely many permanent e-markers are placed on initial segments of X.

(3) There are only finitely many permanent markers placed on initial segments of X.

THE CUPPING PROPERTY

A degree $\boldsymbol{a}$ satisfies the **cupping property** if, for all $\boldsymbol{b} > \boldsymbol{a}$, there exists $\boldsymbol{c} < \boldsymbol{b}$ with $\boldsymbol{a} \vee \boldsymbol{c} = \boldsymbol{b}$.



THE CUPPING PROPERTY

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Background

Theorem [Downey, Jockusch, Stobb]. All a.n.r. degrees satisfy the cupping property.

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Theorem [B,L] The measure of the degrees which satisfy the cupping property is 0. In fact, all 2-random degrees have a strong minimal cover.

We say **b** is a strong minimal cover for **a** if the degrees strictly below **b** are precisely the degrees below and including **a**.

THE CUPPING PROPERTY

Theorem [B,D,L] All degrees below any 2-random degree have a strong minimal cover.

Note that no degree below a 2-random *is* a strong minimal cover.

THE CUPPING PROPERTY

Theorem [Jockusch]. Any non-zero degree below a 2-generic satisfies cupping.

Theorem [Kumabe]. There is a 1-generic with a strong minimal cover.

BEING A MINIMAL COVER

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Background

The most famous result along these lines is the old result of Jockusch:

Theorem [Jockusch]. There is a cone of minimal covers.

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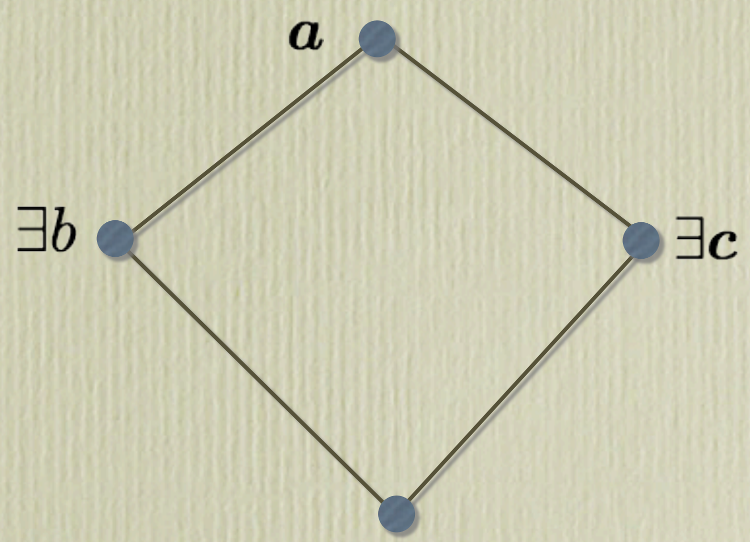
Theorem [Kumabe]. For $n \geq 2$, every n-generic is a minimal cover of an n-generic.

Question. Is every 1-generic a minimal cover?

Question. What is the measure of the degrees which are a minimal cover?

BEING THE TOP OF A DIAMOND

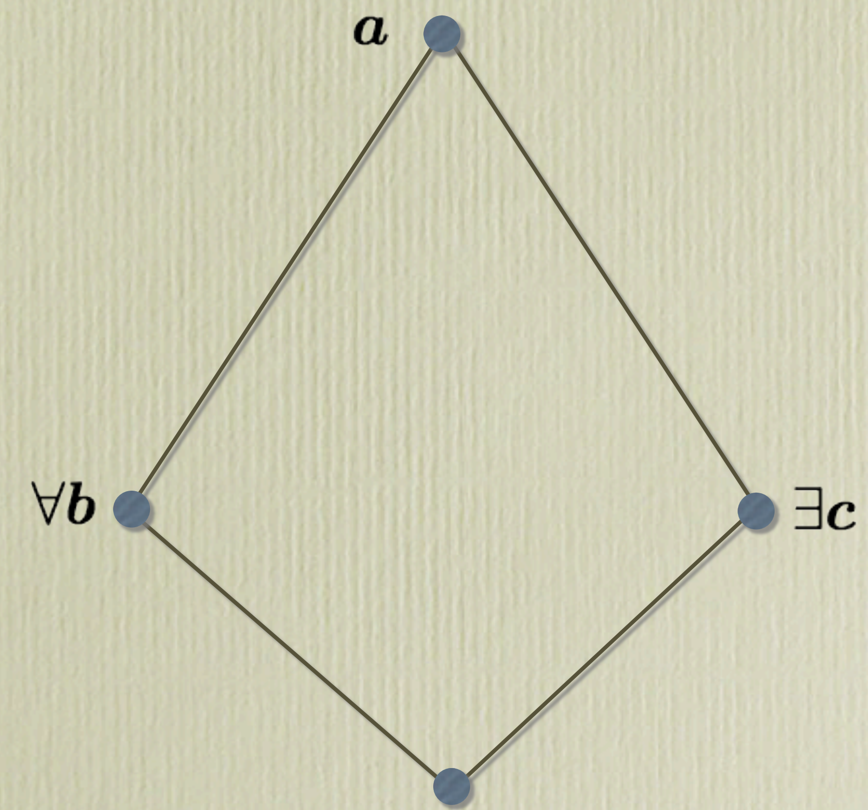
We say that **a** is the top of a diamond if there exist **b** and **c** below **a** which are a minimal pair and join to **a**.



Theorem [B,D,L] Every non-zero degree below a 2-random is the top of a diamond.

THE COMPLEMENTATION PROPERTY

We say $\boldsymbol{a}$ satisfies the **complementation** property if, for all non-zero $\boldsymbol{b} < \boldsymbol{a}$, there exists $\boldsymbol{c} < \boldsymbol{a}$ with $\boldsymbol{b} \vee \boldsymbol{c} = \boldsymbol{a}$ and $\boldsymbol{b} \wedge \boldsymbol{c} = \mathbf{0}$.



THE COMPLEMENTATION PROPERTY

Theorem [Posner]. $\mathbf{0}'$ satisfies the complementation property.

Theorem [Greenberg, Montalbán, Shore]. All generalized high degrees satisfy the complementation property.

Theorem [Kumabe]. All 2-generics satisfy the complementation property.

Question. Do all 1-generics satisfy complementation?

Question. What is the measure of the degrees which satisfy complementation?

	1-random	2-random	1- generic	2-generic
Minimal	no	no	no	no
Bound minimals	some	no	some	no
Join	some	yes	yes	yes
Cupping	some	no	some	yes
Minimal cover	some	?	?	yes
Complementation	?	?	?	yes

Thanks for listening