Maximal chains of computable well partial orders

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Wpos, maximal linear extensions, maximal chains

1. **Wpos, maximal linear extensions, maximal chains**
   - Well-partial-orders
   - Maximal linear extensions of wpos
   - Maximal chains of wpos
2. **Computing maximal chains**
   - Computing strongly maximal chains is hard
   - Computing maximal chains is not easy
   - A way of computing maximal chains
3. **Comparison with reverse mathematics**

Well-partial-orders

A partial order $\mathcal{P} = (P, \leq_P)$ is a well partial order (wpo) if for every $f : \mathbb{N} \to P$ there exists $i < j$ such that $f(i) \leq_P f(j)$.

There are many equivalent characterizations of wpos:

- $\mathcal{P}$ is well-founded and has no infinite antichains;
- every sequence in P has a weakly increasing subsequence;
- every nonempty subset of P has a finite set of minimal elements;
- all linear extensions of $\mathcal{P}$ are well-orders.

The reverse mathematics and computability theory of these equivalences has been studied in (Cholak-M-Solomon 2004).

All equivalences are provable in $\mathsf{WKL}_0+\mathsf{CAC}$.

Some examples of wpos

- Finite partial orders
- Well-orders
- Finite strings over a finite alphabet (Higman, 1952)
- Finite trees (Kruskal, 1960)
- Transfinite sequences with finite labels (Nash-Williams, 1965)
- Countable linear orders (Laver 1971, proving Fraïssé's conjecture)
- Finite graphs (Robertson and Seymour, 2004)

The ordering is some kind of embeddability

Closure properties of wpos

- The sum and disjoint sum of two wpos are wpo
- The product of two wpos is wpo
- Finite strings over a wpo are a wpo (Higman, 1952)
- Finite trees with labels from a wpo are a wpo (Kruskal, 1960)
- Transfinite sequences with labels from a wpo which use only finitely many labels are a wpo (Nash-Williams, 1965)

The maximal order type of a wpo

$\mathcal{P}$ is a wpo $\iff$ all linear extensions of $\mathcal{P}$ are well-orders

We denote by $\mathrm{Lin}(\mathcal{P})$ the collection of all linear extensions of $\mathcal{P}$.

Definition

If $\mathcal{P}$ is a wpo, its maximal order type is

$$o(\mathcal{P}) = \sup\{\, \alpha \mid \exists \mathcal{L} \in \mathrm{Lin}(\mathcal{P})\, \alpha = \mathrm{ot}(\mathcal{L}) \,\}.$$

Theorem (de Jongh – Parikh, 1977)

The $\sup$ *in the definition of* $o(\mathcal{P})$ *is actually a* $\max$*,*
i.e. there exists $\mathcal{L} \in \mathrm{Lin}(\mathcal{P})$ *with order type* $o(\mathcal{P})$*.*
In other words, every $\mathcal{I} \in \mathrm{Lin}(\mathcal{P})$ *embeds into* $\mathcal{L}$*.*
$\mathcal{L}$ *is called a maximal linear extension of* $\mathcal{P}$*.*

Computing maximal linear extensions

Theorem (Montalbán, 2007)

Every computable wpo has a computable maximal linear extension.

However there is no hyperarithmetic function
mapping the index of a computable wpo
to the the index of one of its maximal linear extensions.

The height of a well founded partial order

$\mathcal{P}$ is a wpo $\implies$ $\mathcal{P}$ is well founded and all its chains are well-orders

We denote by $\mathrm{Ch}(\mathcal{P})$ the collection of all chains of $\mathcal{P}$.

Definition

If $\mathcal{P}$ is well founded, its height is

$$\mathrm{ht}(\mathcal{P}) = \sup\{\, \alpha \mid \exists \mathcal{C} \in \mathrm{Ch}(\mathcal{P})\, \alpha = \mathrm{ot}(\mathcal{L}) \,\}.$$

We can also define the height of $x \in P$:

$$\mathrm{ht}_{\mathcal{P}}(x) = \sup\{\, \mathrm{ht}_{\mathcal{P}}(y) + 1 \mid y <_P x \,\}$$

so that $\mathrm{ht}(\mathcal{P}) = \sup\{\, \mathrm{ht}_{\mathcal{P}}(x) + 1 \mid x \in P \,\}$.

Wolk's Theorem

Theorem (Wolk 1967)

If $\mathcal{P}$ is a wpo, the $\sup$ *in the definition of* $\mathrm{ht}(\mathcal{P})$ *is actually a* $\max$*,*
i.e. there exists $\mathcal{C} \in \mathrm{Ch}(\mathcal{P})$ *with order type* $\mathrm{ht}(\mathcal{P})$*.*
Such a chain is called a maximal chain of $\mathcal{P}$*.*

Actually $\mathcal{C}$ *can be chosen so that for every* $\alpha < \mathrm{ht}(\mathcal{P})$
there exists $x \in C$ *such that* $\mathrm{ht}_{\mathcal{P}}(x) = \alpha$*.*
Such a chain is called a strongly maximal chain of $\mathcal{P}$*.*

Two questions

In analogy with the Montalbán's result we ask:

Question

If $\mathcal{P}$ is a computable wpo,
how complicated must maximal and strongly maximal chains of $\mathcal{P}$ be?

It follows from previous work that a computable wpo always has a hyperarithmetic strongly maximal chain.

Question

How complicated must any function taking the computable wpo $\mathcal{P}$ to a maximal chain be?

Computing maximal chains

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Computing strongly maximal chains is hard

Theorem

Let $\alpha < \omega_1^{\mathrm{CK}}$.
There exists a computable wpo $\mathcal{P}$ *such that every strongly maximal chain of* $\mathcal{P}$ *computes* $0^{(\alpha)}$.

The main tool

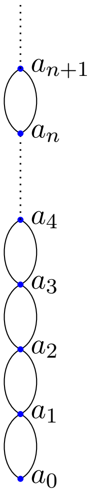
Theorem (Ash-Knight 1990)

Let $\alpha < \omega_1^{\mathrm{CK}}$ and A is a $\mathbf{\Pi}^0_{2\alpha+1}$ set.
There exists a uniformly computable sequence of linear orders $\mathcal{L}_n^A$ such that

$$\mathrm{ot}(\mathcal{L}_n^A) = \begin{cases} \omega^\alpha & \text{if } n \in A; \\ \omega^{\alpha+1} & \text{if } n \notin A. \end{cases}$$

This sequence of linear orderings can be computed uniformly in indices for α as a computable ordinal and A as a $\mathbf{\Pi}^0_{2\alpha+1}$ set.

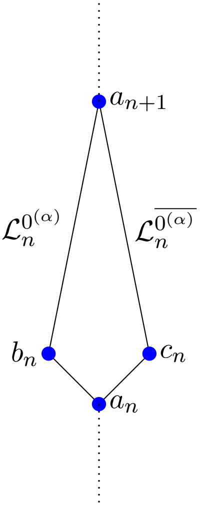
Every strongly maximal chain of $\mathcal{P}$ computes $0^{(\alpha)}$: the global view



Every strongly maximal chain of $\mathcal{P}$ computes $0^{(\alpha)}$: zooming

When $n \in 0^{(\alpha)}$, $\mathrm{ot}(\mathcal{L}_n^{0^{(\alpha)}}) = \omega^\alpha$ and $\mathrm{ot}(\overline{\mathcal{L}_n^{0^{(\alpha)}}}) = \omega^{\alpha+1}$; when $n \notin 0^{(\alpha)}$, $\mathrm{ot}(\mathcal{L}_n^{0^{(\alpha)}}) = \omega^{\alpha+1}$ and $\mathrm{ot}(\overline{\mathcal{L}_n^{0^{(\alpha)}}}) = \omega^\alpha$

$\mathrm{ht}_{\mathcal{P}}(a_n) = \omega^{\alpha+1} \cdot n$ and $\mathrm{ht}(\mathcal{P}) = \omega^{\alpha+2}$



The unique strongly maximal chain $\mathcal{C}$ of $\mathcal{P}$ always picks the $\omega^{\alpha+1}$ side

$n \in 0^{(\alpha)}$ iff $c_n \in \mathcal{C}$

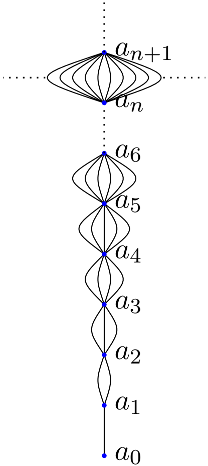
Computing maximal chains is not easy

Theorem

Let $\alpha < \omega_1^{\mathrm{CK}}$*.*
There exists a computable wpo $\mathcal{P}$ *such that* $0^{(\alpha)}$ *does not compute any maximal chain of* $\mathcal{P}$*.*

We are not claiming that the maximal chains of $\mathcal{P}$ compute $0^{(\alpha)}$.

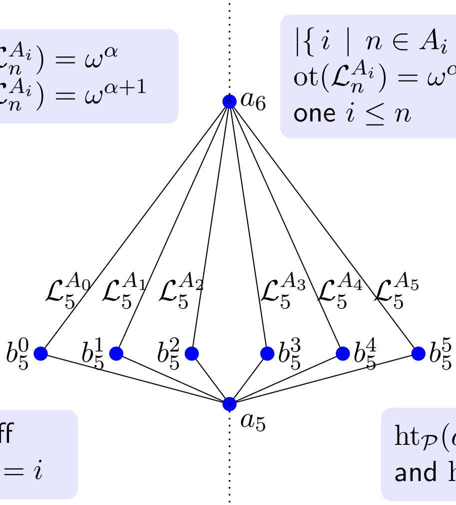
$0^{(\alpha)}$ does not compute any maximal chain of $\mathcal{P}$: the global view



$0^{(\alpha)}$ does not compute any maximal chain of $\mathcal{P}$: zooming

$n \in A_i \implies \mathrm{ot}(\mathcal{L}_n^{A_i}) = \omega^\alpha$
$n \notin A_i \implies \mathrm{ot}(\mathcal{L}_n^{A_i}) = \omega^{\alpha+1}$

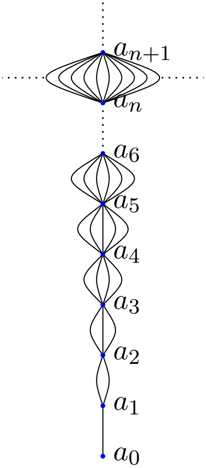
$|\{ i \mid n \in A_i \}| \leq n$ and $\mathrm{ot}(\mathcal{L}_n^{A_i}) = \omega^{\alpha+1}$ for at least one $i \leq n$



where $n \in A_i$ iff $\exists e < n\, \Phi_e^{0^{(\alpha)}}(n) = i$

$\mathrm{ht}_{\mathcal{P}}(a_n) = \omega^{\alpha+1} \cdot n$ and $\mathrm{ht}(\mathcal{P}) = \omega^{\alpha+2}$

$0^{(\alpha)}$ does not compute any maximal chain of $\mathcal{P}$: concluding



Let $\mathcal{C}$ be a maximal chain. Define $\psi \leq_T \mathcal{C}$ by

$$\psi(n) = \begin{cases} i & \text{if } \exists x \in \mathcal{C}\, b_n^i \leq_P x <_P a_{n+1}; \\ \uparrow & \text{otherwise.} \end{cases}$$

Infinitely often ψ picks an $\omega^{\alpha+1}$ chain.

Fix e. There exists $n > e$ such that $n \notin A_{\psi(n)}$.
Thus $\Phi_e^{0^{(\alpha)}}(n) \neq \psi(n)$ and thus $\psi \neq \Phi_e^{0^{(\alpha)}}$.

Therefore $\psi \not\leq_T 0^{(\alpha)}$ and $\mathcal{C} \not\leq_T 0^{(\alpha)}$.

Generic sets for Cohen forcing

Definition

For $\alpha < \omega_1^{\mathrm{CK}}$, a set G is α-generic if the conditions which are initial segments of G suffice to decide all Σ_α-questions.

G is hyperarithmetically generic if it is α-generic for every $\alpha < \omega_1^{\mathrm{CK}}$.

- Almost every set, in the sense of category, is hyperarithmetically generic
- A hyperarithmetically generic is not hyperarithmetic
- A hyperarithmetically generic does not compute any noncomputable hyperarithmetic set

Almost every set computes maximal chains

Theorem

Let G be hyperarithmetically generic.

For every computable wpo $\mathcal{P}$, there exists a maximal chain $\mathcal{C}$ in $\mathcal{P}$ such that $\mathcal{C} \leq_T G$.

If $\mathrm{ht}(\mathcal{P}) < \omega^{\alpha+1}$, then $2 \cdot \alpha$-genericity of G suffices.

- Almost every set, in the sense of category, computes maximal chains
- Every computable wpo has a maximal chain that does not compute any noncomputable hyperarithmetic set, i.e. maximal chains cannot code any $0^{(\alpha)}$

Nonuniformity

Our proof of the previous result has several nonuniform steps.

If $\mathcal{L}_0$ and $\mathcal{L}_1$ are computable well-orders consider $\mathcal{L}_0 \oplus \mathcal{L}_1$, which is a computable wpo.

A maximal chain of $\mathcal{L}_0 \oplus \mathcal{L}_1$ is included in some $\mathcal{L}_i$, and the i is uniformly computable from the maximal chain and the wpo.

Then $\mathcal{L}_{1-i}$ embeds in $\mathcal{L}_i$ and $\mathcal{L}_i$ is the longer chain.

By Ash-Knight this can uniformly code any hyperarithmetic set.

Theorem

There is no hyperarithmetic procedure which calculates a maximal chain of every computable wpo.

Suppose f is such that, for every index e for a computable wpo $\mathcal{P}$, $n \mapsto f(e,n)$ is a maximal chain of $\mathcal{P}$.
Then f computes every hyperarithmetic set.

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Some equivalences with ATR_0

Theorem

Over RCA_0*, the following are equivalent to* ATR_0*:*

1. *the maximal linear extension theorem for wpos [M-Shore 2011];*
2. *the maximal chain theorem for wpos [M-Shore 2011];*
3. *the strongly maximal chain theorem for wpos [M-Shore 2011];*
4. *König's duality theorem for bipartite graphs [Aharoni-Magidor-Shore 1992, Simpson 1994].*

These are all statements of the form $\forall X(\varphi(X) \implies \exists Y\, \psi(X, Y))$.

Different complexity

For statements of the form $\forall X(\varphi(X) \implies \exists Y\, \psi(X,Y))$ we ask
if X is computable, how complicated must Y be?

1. A computable wpos has a computable maximal linear extension
2. A computable wpos has a hyp maximal chain, but maximal chains can be incomparable with all noncomputable hyp sets
3. A computable wpos has a hyp strongly maximal chain, and strongly maximal chains can be of arbitrarily high complexity in the hyp hierarchy
4. There exists a computable bipartite graph such that any pair matching/cover satisfying König's duality computes every hyp set and hence is not hyp

These are four different levels of computational complexity for theorems all axiomatically equivalent to ATR_0.
The phenomena in 2 seems to be new.