Four Related Questions

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- Even better, the degrees that compute a minimal degree have measure zero (Paris).
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- The packing dimensions of the set of minimal Turing degrees is 1 (Downey, Greenberg).

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Proposition (Greenberg and M.)

There is a computable order function $h\colon \omega \to \omega \smallsetminus \{0, 1\}$ such that every h-bounded DNC function computes a real of effective Hausdorff dimension 1.

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We would actually need this in a partially relativized form:

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For an oracle X, is there an h-bounded function that is DNC relative to X and has minimal degree?

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Question 2^X implies that $\dim_H(\text{Minimal}) = 1$.

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Facts (Greenberg and M.)

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Also:

- (Downey, Greenberg, Jockusch, Milans) There is no uniform way to compute a Kurtz random from a DNC_3 function.
- (Greenberg, M.; Khan, M.) For any computable order function h, there is an h-bounded DNC that computes no Kurtz random.

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Fix $k \geqslant 3$. Is there a functional Γ such that $\emptyset <_T \Gamma^f <_T f$ for every DNC_k function $f\colon \omega \to k$?

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It is not hard to see that DNC_k functions are non-minimal, but no uniform proof is known.

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If $f\colon 17^{\omega} \to 2^{\omega}$ is continuous, is f either

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- 17 is an arbitrary number (greater than 3).

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It should be noted that:

Kumar, private communication

There is a continuous $f\colon [0,1] \to \mathbb{R}$ such that

1. f is non-injective on every positive measure set, and
2. f is non-constant on every positive measure set.