Copy vs Diagonalize

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The Game: Copy vs Diag

Let $\mathbb{K}$ be a class of structures on a relational language.

The players: C and D.

They build sequences of finite structures alternatively:

Player D		$\mathcal{D}[0]$	$\subseteq$	$\mathcal{D}[1]$	$\subseteq$	$\cdots$		let $\mathcal{D} = \bigcup_{\mathcal{D}}[s]$
Player C	$\mathcal{C}[0]$	$\subseteq$	$\mathcal{C}[1]$	$\subseteq$	$\mathcal{C}[2]$	$\cdots$		let $\mathcal{C} = \bigcup_{\mathcal{C}}[s]$

note 1: It's allowed to repeat previous play (ex: $\mathcal{C}[s+1] = \mathcal{C}[s]$).

- If $\mathcal{C}, \mathcal{D} \in \mathbb{K}$ are isomorphic then C wins.
- If $\mathcal{C}, \mathcal{D} \in \mathbb{K}$ are not isomorphic then D wins.
- If $\mathcal{C} \in \mathbb{K}$, but $\mathcal{D} \notin \mathbb{K}$, then C wins.
- If $\mathcal{D} \in \mathbb{K}$, but $\mathcal{C} \notin \mathbb{K}$, then D wins.
- If $\mathcal{C} \notin \mathbb{K}$, $\mathcal{D} \notin \mathbb{K}$, then D wins.

note 2: To play a finite structure legally,
a player has to eventually mark it with a move '$\square$'.

Def: $\mathbb{K}$ is *copyable* if C has a computable winning strategy.
$\mathbb{K}$ is *diagonalizable* if D has a computable winning strategy.

Examples

Theorem ([Kach, M])

Linear orderings are diagonalizable.

The ideas in this proof are due to:
[Jockusch, Soare 91]: Not every low linear order has computable copy.
[R. Miller 01]: There is an $\mathcal{L}$ with $Spec(\mathcal{L}) \cap \Delta^0_2 = \Delta^0_2 \setminus \{0\}$.

Theorem

The class $\mathbb{K}$ of Boolean algebras with a predicate for atom, and with infinitely many atoms, is copyable.

The ideas in this proof are due to:

[Downey Jockusch 94]: Every low Boolean algebra has computable copy.

Theorem

The class $\mathbb{K}$ of Boolean algebras with predicates for atom, infinite, atomless, and with infinitely many atoms, is copyable.

The ideas in this proof are due to:

[Thurber 95]: Every low_2 Boolean algebra has low copy.

Copyable impies listable

Def: $\mathbb{K}$ is *computably listable* if
there exists a computable list of all computable structures in $\mathbb{K}$.

Definition

$\mathbb{K}$ is *listable* if there exists a Turing functional Φ, s.t., $\forall X \in 2^\omega$,
Φ^X lists all the X-computable structures in $\mathbb{K}$.

Theorem (M)

If $\mathbb{K}$ is copyable, it's listable.

The theorem doesn't reverse but...

The ∞-Game

Now, player C builds infinitely many structures $\mathcal{C}^0$, $\mathcal{C}^1$, $\mathcal{C}^2$,....

Player D		$\mathcal{D}[0]$	$\subseteq$	$\mathcal{D}[1]$	$\subseteq$	$\cdots$	let $\mathcal{D} = \bigcup_s \mathcal{D}[s]$
	$\mathcal{C}^0[0]$	$\subseteq$	$\mathcal{C}^0[1]$	$\subseteq$	$\mathcal{C}^0[2]$	$\cdots$	let $\mathcal{C}^0 = \bigcup_s \mathcal{C}^0[s]$
Player C			$\mathcal{C}^1[0]$	$\subseteq$	$\mathcal{C}^1[1]$	$\cdots$	let $\mathcal{C}^1 = \bigcup_s \mathcal{C}^1[s]$
					$\mathcal{C}^2[0]$	$\cdots$	let $\mathcal{C}^2 = \bigcup_s \mathcal{C}^1[s]$
						$\vdots$	

- If $\mathcal{D}, \mathcal{C}^0, \mathcal{C}^1, ... \in \mathbb{K}$, and for all i, $\mathcal{D} \not\cong \mathcal{C}^i$ then D wins.
- If $\mathcal{D}, \mathcal{C}^0, \mathcal{C}^1, ... \in \mathbb{K}$, and for some i, $\mathcal{D} \cong \mathcal{C}^i$, then C wins.
- If for some i, $\mathcal{C}^i \notin \mathbb{K}$, then D wins.
- If for all i, $\mathcal{C} \in \mathbb{K}$, but $\mathcal{D} \notin \mathbb{K}$, then C wins.

Def: $\mathbb{K}$ is *∞-copyable* if C has a computable winning strategy.
$\mathbb{K}$ is *∞-diagonalizable* if D has a computable winning strategy.

Theorem (M)

$\mathbb{K}$ is listable if and only if it's ∞-copyable.

Player C now builds infinitely many structures $\mathcal{C}^0$, $\mathcal{C}^1$,....
but needs to choose a single one, $\mathcal{C}^j$, using k-jumps.

Player D		$\mathcal{D}[0]$	$\subseteq$	$\mathcal{D}[1]$	$\subseteq$	$\cdots$	let $\mathcal{D} = \bigcup_s \mathcal{D}[s]$
	e		$f(0)$		$f(1)$	$\cdots$	defining $f : \omega \to \omega$
	$\mathcal{C}^0[0]$	$\subseteq$	$\mathcal{C}^0[1]$	$\subseteq$	$\mathcal{C}^0[2]$	$\cdots$	let $\mathcal{C}^0 = \bigcup_s \mathcal{C}^0[s]$
Player C			$\mathcal{C}^1[0]$	$\subseteq$	$\mathcal{C}^1[1]$	$\cdots$	let $\mathcal{C}^1 = \bigcup_s \mathcal{C}^1[s]$
					$\mathcal{C}^2[0]$	$\cdots$	let $\mathcal{C}^2 = \bigcup_s \mathcal{C}^1[s]$
						$\vdots$	

- If $\{e\}^{f^{(k)}}(0)\uparrow$, then D wins, otherwise, let $j = \{e\}^{f^{(k)}}(0)$.
- If $\mathcal{D}, \mathcal{C}^0, \mathcal{C}^1, \ldots \in \mathbb{K}$, and $\mathcal{D} \not\cong \mathcal{C}^j$, then D wins.
- If $\mathcal{D}, \mathcal{C}^0, \mathcal{C}^1, \ldots \in \mathbb{K}$, and $\mathcal{D} \cong \mathcal{C}^j$, then C wins.
- If for some i, $\mathcal{C}^i \notin \mathbb{K}$, then D wins.
- If for all i, $\mathcal{C} \in \mathbb{K}$, but $\mathcal{D} \notin \mathbb{K}$, then C wins.

Def: $\mathbb{K}$ is *k-copyable* if C has a computable winning strategy.
$\mathbb{K}$ is *k-diagonalizable* if D has a computable winning strategy.

The $0^{(k)}$-Game

Obs:

copyable $\Longrightarrow$ 1-copyable $\Longrightarrow$ 2-copyable $\Longrightarrow$ $\cdots$ ∞-copyable.

diagonalizable $\Leftarrow$ 1-diagonalizable $\Leftarrow$ 2-diagonalizable $\Leftarrow$ $\cdots$ ∞-diagonalizable.

Theorem

Boolean algebras with predicate for atom *are 1-diagonalizable and 2-copyable.*

This can be used to prove:
[M]: Not every low BA is $0^{(2)}$-isomorphic to a computable one.
Recall:
[Downey, Jockusch 94]: Every low BA is $0^{(3)}$-iso. to a computable one.

Theorem

[M]: Linear orderings are 4-copyable.
[Kach, M]: Linear orderings are 2-diagonalizable.

Question Are linear orderings 3-diagonalizable?

The Jump of a structure

Def: The *jump of a structure* $\mathcal{A}$ is another structure $\mathcal{A}'$
built by adding relations $\mathcal{A}$, one for each Σ^c_1-formula.

Example:

- For a linear ordering, $\mathcal{L}' \equiv (\mathcal{L}, succ, 0')$.
- For a Boolean alg. $\mathcal{B}' \equiv (\mathcal{B}, atom, 0')$.
- For a Boolean alg. $(\mathcal{B}, atom)' \equiv (\mathcal{B}, atom, infinite, atomless, 0')$.
- For a vector space $\mathcal{V}' \equiv (\mathcal{V}, LinDep, 0')$.

Thm:[Soskov][M 09] $Spec(\mathcal{A}') = \{\mathbf{x}' : \mathbf{x} \in Spec(\mathcal{A})\}$.

Def: For a class of structures $\mathbb{K}$, let $\mathbb{K}' = \{\mathcal{A}' : \mathcal{A} \in \mathbb{K}\}$

The low property

Definition

We say that $\mathcal{A}$ has the *low property* if, $\forall X, Y \in 2^\omega$ with $X' \equiv_T Y'$,
$\mathcal{A}$ has an X-computable copy $\iff$ $\mathcal{A}$ has a Y-computable copy.
$\mathbb{K}$ has the *low property* if every $\mathcal{A}$ in $\mathbb{K}$ does.

Thm:[M 09] $\mathcal{A}$ has the low property if and only if, $\forall X \in 2^\omega$,
$\mathcal{A}'$ has an X'-computable copy $\iff$ $\mathcal{A}$ has a X computable copy.

Theorem (M)

Assume that $\mathbb{K}$ is Π^c_2-axiomatizable, then if $\mathbb{K}$ has the low property, $\mathbb{K}'$ is listable.

Low Boolean algebras

Let $\mathbb{BA}$ be the class of Boolean algebras

Example [Downey, Jockusch 95][Thurber 95][Knight, Stob 00]
$\mathbb{BA}$, $\mathbb{BA}'$, $\mathbb{BA}''$ and $\mathbb{BA}'''$ have the low property,
and hence $\mathbb{BA}$ has the low_4 property.

Question: Does $\mathbb{BA}^{(n)}$ have the low property for all n?

Theorem (Harris–M)

There is a low_5 BA not $0^{(7)}$-isomorphic to any computable one.

Ideas in the proof:
Let $\mathbb{K}$ be the class of structures $(\mathcal{B}, atom, P)$ where $\mathcal{B} \in \mathbb{BA}$, and P is a unary relation that defines a c.e. subset of the atoms.

- Then $\mathbb{K}$ is 2-diagonalizable.
- $\mathbb{K}$ embeds, in a sense, in $\mathbb{BA}^{(5)}$.

Effectiveness conditions on a class $\mathbb{K}$.

Definition

We say that $\mathbb{K}$ has a *computable 1-back-and-forth structure* if there is effective listing $t_1, t_2, \ldots$ of all the Σ_1-types realized in $\mathbb{K}$, and the set $\{\langle i, j\rangle : t_i \subseteq t_j\}$ is computable.

Example: The following class of structures have computable 1-back-and-forth structures:

- linear orderings,
- Boolean algebras,
- $\mathbb{Q}$-vector spaces,
- equivalence structures.

The main theorem

Theorem (M)

Let $\mathbb{K}$ be a Π^c_2-axiomatizable class of structures with a computable 1-back-and-forth structure.

The following are equivalent:

- $\mathbb{K}$ *has the low property.*
- $\mathbb{K}'$ *is listable.*
- $\mathbb{K}'$ *is* ∞*-copyable.*