The Stable Ramsey's Theorem for Pairs

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Subsystems of Second Order Arithmetic: RCA_0

Definition

A *model* $\mathfrak{M}$ *of second-order arithmetic* consists of a structure $\mathfrak{N}$ for first-order arithmetic, called the *numbers* of $\mathfrak{M}$, and a collection of subsets of $\mathfrak{N}$, called the *reals* of $\mathfrak{M}$.

Definition

RCA_0 is the second-order theory formalizing the following.

- P^-, the axioms for the nonnegative part of a discretely ordered ring.
- $I\Sigma_1$, for φ a Σ^0_1 predicate, if 0 is a solution to φ and the solutions to φ are closed under successor, then φ holds of all numbers.
- The reals are closed under join and relative Δ^0_1-definability.

In an *ω-model* $\mathfrak{M}$, $\mathfrak{N} = \mathbb{N}$ and the reals of $\mathfrak{M}$ form an ideal in the Turing degrees.

Ramsey's Theorem

Definition

For $X \subseteq \mathbb{N}$, let $[X]^n$ denote the size n subsets of X. For $n, m > 0$ and $F : [\mathbb{N}]^n \to \{0, \dots, m-1\}$, $H \subseteq \mathbb{N}$ is *homogeneous for* F iff F is constant on $[H]^n$.

Theorem (Ramsey, 1930)

For all $n, m > 0$ *and all* $F : [\mathbb{N}]^n \to \{0, \dots, m-1\}$, *there is an infinite set* H *such that* H *is homogeneous for* F.

If we fix n and m, then we represent that instance of Ramsey's Theorem by RT^n_m.

Question

What are the first and second order consequences of RT^n_m*?*

Recursion Theoretic Content of Ramsey's Theorem

Theorem (Jockusch, 1972)

- *There is a recursive partition of* F *of pairs such that there is no* F*-homogeneous set which is recursive in* $0'$. ($RCA_0 \nvdash RT^2_2$)
- *There is a recursive partition* F *of triples such that* $0'$ *is recursive in any infinite* F*-homogeneous set.* ($RT^3_2 \vdash ACA_0$)

Theorem (Seetapun, 1995)

There is an ideal J *in the Turing degrees as follows.*

- $0' \notin J$
- *For every* $F : [\mathbb{N}]^2 \to 2$ *in* J, *there is an infinite* F*-homogeneous* H *in* J.

($RT^2_2 \nvdash ACA_0$)

RT^2_2, second order consequences

Definition

- An infinite set X is *cohesive* for a family $R_0, R_1, \dots$ of sets iff for each i, one of $X \cap R_i$ or $X \cap \overline{R_i}$ is finite. COH is the principle stating that every family of sets has a cohesive set.
- A partition $F : [\mathbb{N}]^2 \to \mathbb{N}$ is *stable* iff for all x, $\lim_{y \to \infty} F(x, y)$ exists. SRT^2_2 is the principle RT^2_2 restricted to stable partitions.

Theorem (Cholak, Jockusch, and Slaman, 2001)

$RCA_0 \vdash \left[RT^2_2 \iff (SRT^2_2 \,\&\, COH)\right]$

Stable Partitions

Question

Does SRT^2_2 imply RT^2_2?

Observations:

- If F is a stable partition of pairs, then each x has an *eventual color* given by $\lim_{y \to \infty} F(x, y)$, which can be computed from F'.
- If H is infinite and monochromatic with respect to eventual color, then H can compute an F-homogeneous set. Consequently, F' can compute one.
- SRT^2_2 is equivalent to "For every Δ^0_2 property A, there is an infinite set H contained in or disjoint from A."

Proposal and Rejection

Proposal:

- Show that for every Δ^0_2 subset A of $\mathbb{N}$, there is a low infinite set H contained in or disjoint from A.
- Iterate this fact to build an ideal J in the Turing degrees consisting of only low sets such that for every Δ^0_2 subset A of $\mathbb{N}$, there is an infinite $H \in J$ contained in or disjoint from A.
- Conclude that $(\mathbb{N}, J)$ is a model of SRT^2_2 which is not a model of RT^2_2.

Rejection:

Theorem (Downey, Hirschfeldt, Lempp, and Solomon, 2001)

There is a Δ^0_2 set with no infinite low subset in either it or its complement.

RT^2_2, first order consequences

The following two theorems bracket the first order theory of RT^2_2.

Theorem (Hirst, 1987)

$RCA_0 \vdash (SRT^2_2 \implies B\Sigma_2)$, *where $B\Sigma_2$ is the formalization of the assertion "If a Σ_2 property holds of every element of a finite set then there is a uniform bound on the witnesses."*

Theorem (Cholak, Jockusch, and Slaman, 2001)

RT^2_2 is conservative over $RCA_0 + I\Sigma_2$ for Π^1_1-sentences.

Question

Does either of SRT^2_2 or RT^2_2 imply $I\Sigma_2$?

SRT^2_2 in a Customized Model

Theorem (Chong, Slaman, and Yang)

There is a model $\mathfrak{M}$ of RCA_0 with the following properties.

- $\mathfrak{M} \models SRT^2_2$
- $\mathfrak{M} \models \neg I\Sigma_2$
- *Every real in $\mathfrak{M}$ is low in $\mathfrak{M}$.*

Corollary

SRT^2_2 proves neither $I\Sigma_2$ nor RT^2_2 over RCA_0.

The Natural Numbers of $\mathfrak{M}$

We construct a model $\mathfrak{N}$ of $P^- + B\Sigma_2$ within which there is a function g that is recursive in $0'$ behaving as follows.

- $\mathfrak{N}$ is the union of a sequence of Σ_1-elementary end-extensions of models of PA.

$$\mathfrak{N}_1 \prec_{\Sigma_1,e} \mathfrak{N}_2 \prec_{\Sigma_1,e} \mathfrak{N}_3 \prec_{\Sigma_1,e} \cdots \prec_{\Sigma_1,e} \mathfrak{N}$$

- For each $i \in \mathbb{N}$, $g(i) \in \mathfrak{N}_i \setminus \mathfrak{N}_{i-1}$, hence $\mathfrak{N} \not\models I\Sigma_2$.
- For each set $Y \subseteq \mathbb{N}$, if Y is definable in $\mathfrak{N}$ then Y has an $\mathfrak{N}$-finite end-extension in $\mathfrak{N}$.
- $\mathfrak{N}$ is countable.

Low Subsets of Δ^0_2 Sets in $\mathfrak{N}$

Suppose that A is Δ^0_2 in $\mathfrak{N}$. Build $H_0 \subseteq A$ and $H_1 \subseteq \overline{A}$.

- For $a \in \mathfrak{N}$, adapt Seetapun's argument so as to decide $H_0' \upharpoonright a$ or of $H_1' \upharpoonright a$.
- Use reflection to the models $\mathfrak{N}_i$ of PA to show that the activity of a single step is bounded in $\mathfrak{N}$.
- Construct H_0 and H_1 by an ω-length recursion. Each step has two parts:
  - an $\mathfrak{N}$-finite extension, uniformly computed from $0'$
  - a global constraint, non-uniformly computed from $0'$, depending on a Σ_2-boolean condition
- Conclude that the construction of H_0 and H_1 is Δ^0_2 in $\mathfrak{N}$, using the $\mathfrak{N}$-finite parameter extending the sequence of Σ_2-boolean conditions that appear in the construction.

Extending to a Model of SRT^2_2

We obtain an H as desired, but it does not preserve the Σ_1-reflection properties of $\mathfrak{N}$ that were used to construct it. So, we cannot simply iterate the argument.

We use a full-approximation construction to obtain a collection of subsets $\mathfrak{J}$ of $\mathfrak{N}$ with the desired properties:

- $(\mathfrak{N}, \mathfrak{J}) \models RCA_0$
- For any $X \in \mathfrak{J}$, X is low in $\mathfrak{N}$.
- For every A which is Δ^0_2 in $\mathfrak{N}$, there is an infinite element of $\mathfrak{J}$ which is contained in it or in its complement.

Questions

Question

- *Does "$RCA_0 + SRT^2_2 \nvdash RT^2_2$" settle the issue of whether there is an effective proof of Ramsey's Theorem for Pairs given that Ramsey's Theorem for Pairs is true for stable partitions?*
- *For ω-models, does SRT^2_2 imply RT^2_2? Perhaps $I\Sigma_2$ is sufficient to deduce the implication.*

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