Algorithmic Randomness in Ergodic Theory

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Definition

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We are interested in sets of the form $\bigcup_N V_N$ where the sets V_N are uniformly computably enumerable open sets with some constraint on the sizes of the V_N so that $\mu(\bigcap_N V_N) = 0$.

Definition

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In our case $\Omega = 2^\omega$, $\mathcal{B}$ is the σ-algebra generated by the open sets

$$[\sigma] = \{\sigma^\frown \rho \mid \rho \in 2^\omega\},$$

and μ is the measure generated by

$$\mu([\sigma]) = \frac{1}{2^{-|\sigma|}}.$$

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That is, what is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(T^i x)?$$

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Theorem (Birkhoff Ergodic Theorem)

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x is a *Birkhoff point* for a transformation T with respect to a collection of sets if for every set A in this collection,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(T^i x)$$

converges.

Question

What is the relationship between

- *Being a Birkhoff point for some family of transformations and sets, and*
- *Satisfying notions of algorithmic randomness?*

We focus on two natural domains for varying this problem:

- On the computability side, *how computable are the sets A we consider*?
- On the ergodic theory side, *how nice is the transformation T*?

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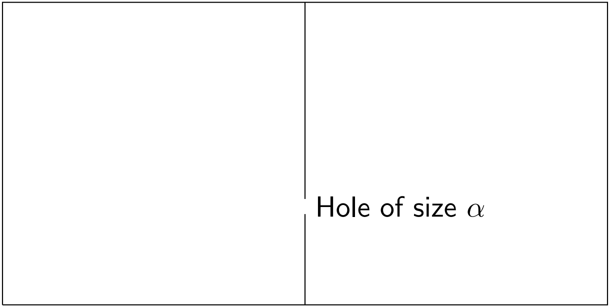
- On the computability side, *how computable are the sets A we consider*?
- On the ergodic theory side, *how nice is the transformation T*?

We consider the cases where A is either computable or computably enumerable. The case where A is computably enumerable but $\mu(A)$ is computable is generally very similar to the computable case, through the proofs are slightly more complicated.

A point is a Birkhoff point for the given family of transformations and sets if and only if it is ____________-random.

Transformation: Computability of sets:	Arbitrary	Ergodic
Computable		
Σ^0_1/Π^0_1		

In general, the rate of convergence is *not* computable. α is a non-negative real number close to 0 which may or may not actually be 0.



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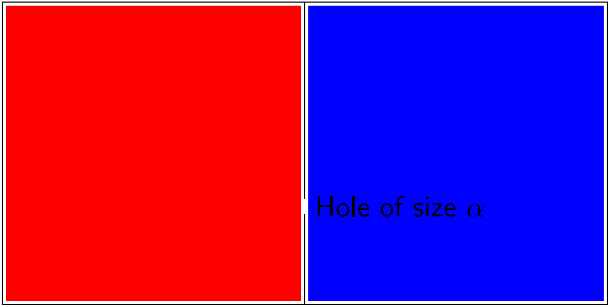

Definition

A dynamical system is *ergodic* if any of the following equivalent conditions hold:

- For every set A and almost every point x, the ergodic average

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(T^i x) = \mu(A),$$

- For all sets A, B with positive measure there is an n such that $\mu(A \cap T^n B) > 0$,
- Whenever $T(A) = A$, either $\mu(A) = 0$ or $\mu(A) = 1$,
- If $\mu(A) > 0$ then $\mu(\bigcup_n T^n A) = 1$.

Definition

Write $A_n(x) = \frac{1}{n}\sum_{i=0}^{n-1}\chi_A(T^i x)$.

Theorem (Avigad-Gerhardy-T.)

In an ergodic computable dynamical system, there is a computable function $n(\epsilon)$ such that

$$\mu(\{x \mid \max_{n(\epsilon)\leq m\leq k}\left|A_m(x) - A_{n(\epsilon)}(x)\right| > \epsilon\}) < \epsilon.$$

Theorem (Gacs, Hoyrup, and Rojas)

Suppose x is Schnorr random. Then x is a Birkhoff for computable sets in computable ergodic systems.

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In fact, the transformation constructed is much stronger than merely ergodic.

A point is a Birkhoff point for the given family of transformations and sets if and only if it is ______________-random.

Transformation: Computability of sets:	Arbitrary	Ergodic
Computable		Schnorr
Σ^0_1/Π^0_1		

Theorem (Bienvenu-Day-Hoyrup-Mezhirov-Shen/Franklin-Greenberg-J. Miller-Ng)

Suppose x is Martin-Lof random. Then x is a Birkhoff point for Σ^0_1 sets in computable ergodic systems.

Theorem (Bienvenu-Day-Hoyrup-Mezhirov-Shen/Franklin-Greenberg-J. Miller-Ng)

Suppose x is Martin-Lof random. Then x is a Birkhoff point for Σ_1^0 sets in computable ergodic systems.

Theorem (Bienvenu-Day-Hoyrup-Mezhirov-Shen)

Suppose x is not Martin-Lof random. Then there is a computable ergodic dynamical system and a Σ_1^0 set A such that the ergodic average at x does not converge to the value $\mu(A)$.

A point is a Birkhoff point for the given family of transformations and sets if and only if it is ______________-random.

Transformation: Computability of sets:	Arbitrary	Ergodic
Computable		Schnorr
Σ^0_1/Π^0_1		Martin-Lof

In the non-ergodic case, we have to use a weaker effective form of the ergodic theorem.

Definition

Let a dynamical system, a set A be given, and $\alpha < \beta$ be given. Let $\upsilon(x)$ be the supremum of those N such that there exist

$$u_1 < v_1 < u_2 < v_2 < \cdots < u_N < v_N$$

such that for all $i \leq N$,

$$A_{u_i}(x) < \alpha < \beta < A_{v_i}(x).$$

Theorem (Bishop)

$\int \upsilon(x) d\mu \leq \frac{\mu(A)(1-\alpha)}{\beta-\alpha}$

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Proof.

Suppose the ergodic average at x does not converge. Then there must be $\alpha < \beta$ so that $\upsilon(x) = \infty$.

$$V_N = \{x \mid \upsilon(x) \geq N\}$$

is computably enumerable, Bishop's Theorem gives a bound on $\mu(V_N)$, and if $\upsilon(x) = \infty$ then $x \in \bigcap_N V_N$. Therefore x is not Martin-Lof random. □

Theorem (Franklin-T.)

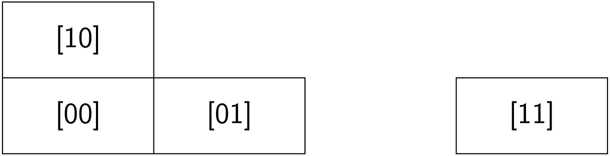
Suppose x is not Martin-Lof random. Then there is a computable dynamical system and a computable set A such that the ergodic average at x does not converge.

Idea of proof: Construct an ad hoc computable dynamical system using *cutting and stacking*.

Cutting and stacking was introduced by Chacon to construct dynamical systems with very specific mixing properties.

2^{ω}

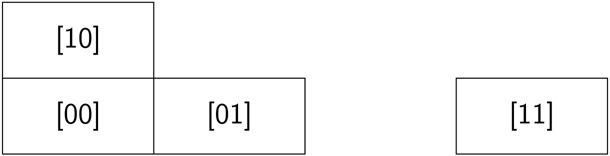
[00]	[01]	[10]	[11]



This means we have decided that for every $\sigma \in 2^\omega$, $T(00^\frown\sigma) = 10^\frown\sigma$.

A problem: because T is supposed to be a computable transformation, we need to arrange that, outside a G_δ measure 0 set, for each n and each $\sigma \in 2^\omega$, there is a k so that $T([\sigma \upharpoonright k]) \subseteq [\tau]$ with $|\tau| = n$.

But sometimes we need to wait, potentially forever, before specifying where a block goes.



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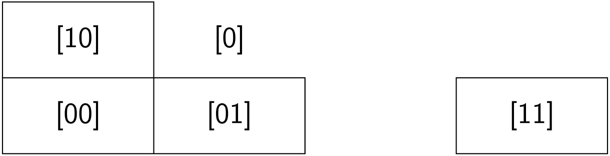
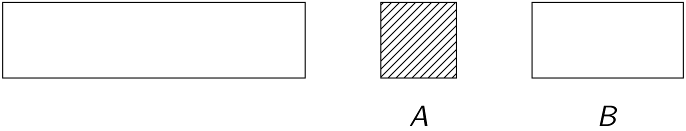


This mens we have decided that $T([01]) \subseteq \sigma[0]$.

Given $x \in 2^\omega$ which is not Martin-Lof random, we we wish to construct a dynamical system in which x is not a Birkhoff point for the ergodic average.

If x is not Martin-Lof random, there is a Martin-Lof test with $x \in \bigcap_N V_n$. We begin by separating a region we know contains x, a computable set A and a computable set B which is disjoint from A and does not contain x.



When we enumerate segments into appropriate V_i, we stack enough intervals from A above that segment to ensure that the ergodic averge gets above $1/2$, and then stack intervals from B to bring the average back below $1/3$.

A point is a Birkhoff point for the given family of transformations and sets if and only if it is _____________-random.

Transformation: Computability of sets:	Arbitrary	Ergodic
Computable	Martin-Löf	Schnorr
Σ^0_1/Π^0_1		Martin-Lof

Theorem (Bishop)

$\int \upsilon_A(x) d\mu \leq \frac{\mu(A)(1-\alpha)}{\beta-\alpha}$ *where* $\upsilon_A(x) \geq N$ *means there are*

$$u_1 < v_1 < u_2 < v_2 < \cdots < u_N < v_N$$

such that for all $i \leq N$,

$$A_{u_i}(x) < \alpha < \beta < A_{v_i}(x).$$

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Question

If $\mu(C)$ is small, what can we say about

$$\int \max\{\upsilon_A(x), \upsilon_{A\cup C}(x)\} d\mu?$$

Definition

Let $A \subseteq B$ be given. $\tau_{A,B}(x)$ is the largest N such that there are

$$u_1 < v_1 < u_2 < v_2 < \cdots < u_N < v_N$$

such that for all $i \leq N$,

$$A_{u_i}(x) < \alpha < \beta < B_{v_i}(x).$$

Theorem (Franklin-T.)

If $\mu(B \setminus A) < \epsilon$, there is a set W with $\mu(W) < \frac{4\epsilon}{\beta - \alpha}$ such that

$$\int_{\Omega \setminus W} \tau_{A,B}(x) d\mu$$

is finite.

Theorem (Franklin-T.)

Suppose x is weakly-2-random. Then x is a Birkhoff point for Σ^0_1 sets in arbitrary computable systems.

A point is a Birkhoff point for the given family of transformations and sets if and only if it is _____________-random.

Transformation: Computability of sets:	Arbitrary	Ergodic
Computable	Martin-Lof	Schnorr
Σ_1^0/Π_1^0	$\leq$weakly-2-random	Martin-Lof