Balancing Randomness

Daniel Turetsky, Victoria University of Wellington

http://homepages.ecs.vuw.ac.nz/∼dan/

(Joint with Laurent Bienvenu, Noam Greenberg, Antonín Kučera and André Nies)

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Theorem (Hirschfeldt, Nies & Stephan)

Every c.e. set below an incomplete MLR is K-trivial.

Question

Is every K-trivial computable from an incomplete MLR?

Theorem (Kučera & Slaman)

There is a low degree which bounds all K-trivials.

Question

Is there a low MLR which computes every K-trivial?

Theorem

There is a K-trivial A such that all MLR $X >_T A$ are super-high ($X' \geq_{tt} \emptyset''$).

Suppose f is a nondecreasing function. Then the derivative exists almost everywhere. So if we effectivize f, some form of randomness implies $f'(z)$ exists.

Suppose M is a martingale. Then $\lim_n M(X \upharpoonright_n)$ exists almost everywhere. So if we effective M, some form of randomness implies $\lim_n M(X \upharpoonright_n)$ exists.

We effectivize nondecreasing functions with interval c.e. functions:

Definition

An increasing function f is *interval c.e.* if for all rationals $a < b$, $f(b) - f(a)$ is uniformly left c.e..

Example: the variation of a computable function is interval c.e..

We effectivize martingales with left c.e. martingales. (Not supermartingales.)

Question

For which reals Z does $f'(Z)$ exist for all non decreasing, interval c.e. functions f?

Question

For which reals Z does every left c.e. martingale converge?

Definition

A *balanced test* is a nested sequence of Σ^0_1 classes $\langle \mathcal{U}_n \rangle$ with $\lambda(\mathcal{U}_n) \leq 2^{-n}$, and the indices of the $\mathcal{U}_n$ are given by a computable approximation function which changes at most 2^n times on input n.

A real Z passes $\langle \mathcal{U}_n \rangle$ if $Z \notin \bigcap \mathcal{U}_n$.

A real Z is *balanced random* if it passes every balanced test.

Weak Demuth random $\Rightarrow$ balanced random $\Rightarrow$ difference random (all implications proper)

Theorem

If Z is balanced random, then $f'(Z)$ exists for all nondecreasing, interval c.e. functions f.

Idea of proof (for martingale version):
Fix rationals $\alpha < \beta$ such that $\exists^\infty n\ M(Z\restriction_n) < \alpha$, $\exists^\infty n\ M(Z\restriction_n) > \beta$.

An *upcrossing* is a pair of strings $\sigma \prec \tau$ with $M(\sigma) \leq \alpha$, $M(\tau) > \beta$.

Create $\mathcal{U}_n$ by searching for upcrossings.

Every time $M(\langle\rangle)$ passes a multiple of 2^{-n}, change the version of $\mathcal{U}_n$.

Definition

For a Π^0_1 class $\mathcal{C} \subseteq \mathbb{R}$ and an interval $I \subseteq \mathbb{R}$, *the density of $\mathcal{C}$ in I* is $\rho_{\mathcal{C}}(I) = \lambda(I \cap \mathcal{C})/\lambda(I)$.

For $Z \in \mathbb{R}$, *the density of Z in $\mathcal{C}$* is $\rho_{\mathcal{C}}(Z) = \liminf_{\substack{|I| \to 0 \\ Z \in I}} \rho_{\mathcal{C}}(I)$.

(Note that $\rho_{\mathcal{C}}([\sigma])$ is a right c.e. martingale.)

By Lebesgue density, almost every $Z \in C$ has density 1 in $\mathcal{C}$.

Theorem (Bienvenu, Hölzl, Miller, Nies)

If Z is MLR and not LR-hard, then for every Π^0_1 $\mathcal{C} \ni Z$, $\rho_{\mathcal{C}}(Z) = 1$.

If $f'(Z)$ exists for all non-decreasing, interval c.e. functions f, then $\rho_{\mathcal{C}}(Z) = 1$ for all Π^0_1 $\mathcal{C} \ni Z$.

So if Z is balanced random, then $\rho_{\mathcal{C}}(Z) = 1$ for all Π^0_1 $\mathcal{C} \ni Z$.

Does this tell us anything new? Is there an LR-hard balanced random?

If $f'(Z)$ exists for all non-decreasing, interval c.e. functions f, then $\rho_{\mathcal{C}}(Z) = 1$ for all Π^0_1 $\mathcal{C} \ni Z$.

So if Z is balanced random, then $\rho_{\mathcal{C}}(Z) = 1$ for all Π^0_1 $\mathcal{C} \ni Z$.

Does this tell us anything new? Is there an LR-hard balanced random?

We couldn't build one.

As Carl mentioned, when you have problems, define them away.

The problem was that $\mathcal{U}_n$ might use a lot of its changes, while $\mathcal{U}_{n+1}$ still has most of its changes left.

Definition

An *Oberwolfach test* is a balanced test such that every time $\mathcal{U}_{n+1}$ changes twice, $\mathcal{U}_n$ changes at least once.

A real is *Oberwolfach random* if it passes every Oberwolfach test.

Balanced random $\Rightarrow$ Oberwolfach random $\Rightarrow$ difference random

Theorem

If Z is Oberwolfach random, then $f'(Z)$ exists for all nondecreasing interval c.e. functions f.

Does this fix the problem? No.

But it gets us some nice properties:

Theorem

If Z is MLR but not Oberwolfach random, then Z is h-JT-hard for any computable order h with $\sum_x \frac{1}{h(x)} < \infty$, and hence Z is super-high.

Theorem

There is a K-trivial c.e. set A which is not below any Oberwolfach random.

Proof sketch (for balanced random):

Let $\mathcal{E} = \{Y \mid \exists x \in A[\Theta^Y(x) = 0]\}$.

Beginning at stage s_0, pick a $v \notin A$, and let
$\mathcal{U}_n = \{Y \notin \mathcal{E}_{s_0} \mid A\restriction_{v+1} \prec \Theta^Y\}$.

When $\lambda(\mathcal{U}_n)$ reaches 2^{-n}...

1. if $\lambda(\mathcal{U}_n \cap \mathcal{E}_s) > 2^{-(n+1)}$, choose a large v and start over.
2. if $c_K(v, s) > 2^{-(n+1)}$, choose a large v and start over.
3. otherwise, enumerate v into A, choose a large v and start over.

Putting these two together,

Corollary

There is a K-trivial A such that all $X >_T A$ which are MLR are super-high.

Also,

Theorem (Figueira, Hirschfeldt, Miller, Ng, Nies)

If X is not balanced random, then every $Y \in MLR^X$ is balanced random.

Corollary

If $X \oplus Y$ is MLR, then at least one of X, Y is balanced random.

Corollary

There is a K-trivial which is not below both halves of a MLR.

Question

Can we separate Oberwolfach randomness from difference randomness?