Isolation: Motivations and Applications

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Two results from Kaddah's thesis

1. Every low c.e. degree is branching in the d.c.e. degrees.

2. There are two d.c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ such that they have infimum $\mathbf{d}$ in the d.c.e. degrees, and there also exists a 3-c.e. degree $\mathbf{x}$ such that

$$\mathbf{d} < \mathbf{x} < \mathbf{a}, \mathbf{b}.$$

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This can be generalized as:

For $n \geq 2$, there are two n-c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ such that they have infimum $\mathbf{d}$ in the n-c.e. degrees, and there also exists an $(n+1)$-c.e. degree $\mathbf{x}$ such that

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Consequences:

Cooper and Yi's definition

Definition - Cooper and Yi 95

A d.c.e. degree **d** is isolated by a c.e. degree **a** if $\mathbf{a} < \mathbf{d}$ is the greatest c.e. degree below **d**.

A d.c.e. degree **d** is isolated, if it is isolated by some c.e. degree **a**.

1. Generalize this to isolated $(n+1)$-c.e. degrees.

2. Kaddah's work implies such isolated degrees.

A direct construction

- The construction of isolated degrees is a "wait and catch an error" process:

  construct a d.c.e. set D and a c.e. set A such that

- $D \not\leq_T A$
- each c.e. set W reducible to $A \oplus D$ is also reducible to A (via a p.c. functional Γ constructed by us).

Weak density results

Theorem - Ding and Qian 96; LaForte 96; Arslanov, Lempp and Shore 96

Both the isolated d.c.e. degrees and the nonisolated d.c.e. degrees are dense in the c.e. degrees.

Kaddah's branching theorem, again - each low nonbranching c.e. degree is isolating.

Theorem - Ishmukhametov and Wu; Li, Wu and Yang

There is a high d.c.e. degree isolated $\mathbf{d}$ by a low c.e. degree $\mathbf{c}$.

Such a c.e. degree $\mathbf{c}$ can be found below any nonzero c.e. degree $\mathbf{a}$.

Nonisolating degrees

Theorem - Arslanov, Lempp, Shore 96

The nonisolating degrees are downwards dense in the d.c.e. degrees, and can occur in every jump class.

Theorem - Salts 2000

The nonisolating degrees are not dense in the c.e. degrees.

Note that the interval of isolating degrees are downwards dense in the c.e. degrees.

Bubbles

Arslanov, Kalimullin, Lempp proved recently that $\mathcal{D}_2$ and $\mathcal{D}_3$ are not elementarily equivalent, by showing that $\mathbf{D}_3$ contains 3-bubbles, but not $\mathbf{D}_2$.

This implies the following bubble theorem:

Theorem - Arslanov, Kalimullin, Lempp

There exist a d.c.e. degrees $\mathbf{e}$ and $\mathbf{d}$, such that $\mathbf{0} < \mathbf{d} < \mathbf{e}$ and any d.c.e. degree $\mathbf{u} \leq \mathbf{e}$ is comparable with $\mathbf{d}$.

$\mathbf{d}$ above should be c.e. and hence $\mathbf{d}$ isolates $\mathbf{e}$.

Bubbles are isolation pairs

Theorem - Arslanov, Kalimullin, Lempp

Let D and E be d.c.e. sets with $E \not\leq_T D$, and X be a c.e. set such that

- both D and E are c.e. in X
- $D \not\leq_T X \leq_T E$.

Then there exists a d.c.e. set U with $X \leq_T U \leq_T E$ but U and D are Turing incomparable.

$\mathbf{d}$ above is c.e. and hence $\mathbf{d}$ isolates $\mathbf{e}$.

Bubbles are isolation pairs

Question - Arslanov

Distribution of such isolating degrees - relating to definibility of c.e. degrees in the d.c.e. degrees

Non-Σ_1-substructures

1. $\mathcal{R} \not\preceq_1 \Delta^0_2$. (Slaman, 1983)
2. $\mathcal{R} \not\preceq_1 \mathcal{D}_2$. (Yang & Yu, 2006)
3. $\mathcal{D}_m \not\preceq_1 \mathcal{D}_n$ for $m < n$. (Cai, Shore & Slaman, 2011)

Lachlan's nondiamond theorem

1. Lachlan's nondiamond theorem.
2. Slaman's cupping theorem
3. Slaman Triples

Theorem - Slaman 83

There are c.e. degrees $\mathbf{a}, \mathbf{b}$, $\mathbf{c}$, and a Δ^0_2 degree $\mathbf{d}$ with $\mathbf{0} < \mathbf{d} < \mathbf{a}$ such that

- $\mathbf{a}, \mathbf{b}$, $\mathbf{c}$ form a Slaman triple,
- $\mathbf{d} \vee \mathbf{b} \not\geq \mathbf{c}$.

Theorem - Yang and Yu 06

There are c.e. degrees $\mathbf{a}, \mathbf{b}$, $\mathbf{c}$, $\mathbf{e}$ and a d.c.e. degree $\mathbf{d} < \mathbf{a}$ such that $\mathbf{d} \not< \mathbf{e}$, $\mathbf{d} \vee \mathbf{b} \not\geq \mathbf{c}$ and for any c.e. degree $\mathbf{w} < \mathbf{a}$, either $\mathbf{w} \vee \mathbf{b} \geq \mathbf{c}$ or $\mathbf{w} < \mathbf{e}$.

Theorem - Cai, Shore and Slaman 2012

There are c.e. degrees $\mathbf{a}, \mathbf{b}$, $\mathbf{c}$, $\mathbf{e}$ and an $(n+1)$-c.e. degree $\mathbf{d} < \mathbf{a}$ such that $\mathbf{d} \vee \mathbf{b} \not\geq \mathbf{c}$, $\mathbf{d} \not< \mathbf{e}$, and for any n-c.e. degree $\mathbf{v} < \mathbf{a}$, either $\mathbf{v} \vee \mathbf{b} \geq \mathbf{c}$ or $\mathbf{v} < \mathbf{e}$.

A variant of isolation

e, a c.e. degree, above bounds all the c.e. degrees below **d**, and we say that **d** is isolated by **e**, from side.

Question:

A question of Arslanov on the number of parameters

Is it necessary for such a c.e. degree **e** not below **d**?

1. For Cai-Shore-Slaman, if we move **e** below **d**, then we need **e** be n-c.e.
2. It is nontrivial for isolation from side only when **d** is nonisolated, in the sense of Cooper and Yi.

Three theorems

- Arslanov's cupping theorem: Every nonzero d.c.e. degree cups.
- Downey's diamond theorem: the diamond lattice can be embedded into the d.c.e. degrees preserving 0 and 1.
- Cooper, Harrington, Lachlan, Lempp and Soare's Nondensity Theorem:

  There exists a maximal d.c.e. degree $\mathbf{d} < \mathbf{0}'$, and hence the d.c.e. degrees are not dense.

Isolation and diamond embeddings

Theorem - Wu 02

There are c.e. degrees $\mathbf{a}, \mathbf{c}$ and a d.c.e. degree $\mathbf{d}$ such that $\mathbf{a}, \mathbf{c}$ form a minimal pair, $\mathbf{a}$ isolates $\mathbf{d}$ and $\mathbf{c}$ cups $\mathbf{d}$ to $\mathbf{0}'$.

$\{\mathbf{0}, \mathbf{c}, \mathbf{d}, \mathbf{0}'\}$ is a diamond embedding.

Theorem - Downey, Li and Wu

If $\mathbf{c} > \mathbf{0}$ is cappable, then there are a c.e. degree $\mathbf{a}$ and a d.c.e. degree $\mathbf{d}$ such that $\mathbf{a}$ isolates $\mathbf{d}$ and $\mathbf{c}$ cups $\mathbf{d}$ to $\mathbf{0}'$ and caps $\mathbf{a}$ to $\mathbf{0}$.

As a consequence, a c.e. degree is cappable if and only if it has a complement in the d.c.e. degrees.

Maximal degrees and almost universal cupping property

Say that a d.c.e. degree $\mathbf{d}$ has almost universal cupping property if it cups every c.e. degree not below it to $\mathbf{0}'$.

- The incomplete maximal d.c.e. degree constructed by Cooper, et al. does have this property.
- A direct construction of a d.c.e. degree with this property.

Theorem - Liu and Wu

There is an almost universal cupping d.c.e. degree $\mathbf{d}$, and a c.e. degrees $\mathbf{b} < \mathbf{d}$ such that $\mathbf{d}$ is isolated by a c.e. degree $\mathbf{b}$ below it.

Furthermore, $\mathbf{b}$ can be cappable.

From this, we have strong diamond embeddings.

We construct a d.c.e. set D, a nonrecursive c.e. sets B satisfying the isolation requirements and also the following cupping requirements:

$$\mathcal{R}_e\colon\ K = \Gamma_e^{B,D,W_e} \ \vee\ W_e = \Delta_e^B,$$

where Γ_e and Δ_e are p.c. functionals constructed by us.

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where Γ_e and Δ_e are p.c. functionals constructed by us.

Theorem - Fang, Liu and Wu

For any nonzero cappable c.e. degree $\mathbf{c}$, there is a d.c.e. degree $\mathbf{d}$ with almost universal cupping property and a c.e. degree $\mathbf{b} < \mathbf{d}$ such that $\mathbf{b}$ isolates $\mathbf{d}$ and that $\mathbf{c}$ and $\mathbf{b}$ form a minimal pair.

1. It covers a theorem of Downey, Li and Wu.
2. It also implies Li-Yi's cupping theorem.

Theorem - Li and Yi

There are two incomplete d.c.e. degrees $\mathbf{d}$ and $\mathbf{e}$ such that every nonzero c.e. degree cups at least one of $\mathbf{d}$ and $\mathbf{e}$ to $\mathbf{0}'$.

Questions:

1. Maximal d.c.e. degrees.

2. Decidability of fragments of the theory of d.c.e. degrees

3. Super-minimal pairs.

Thanks!