Uniformly multiply permitting c.e. sets and lattice embeddings into the c.e. degrees

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(Joint work with Nadine Losert)

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Basic Papers on Multiple Permitting

- (DJS1990) Downey, Rod; Jockusch, Carl; Stob, Michael. Array nonrecursive sets and multiple permitting arguments. Recursion theory week (Oberwolfach, 1989), 141–173, Lecture Notes in Math., 1432, Springer, Berlin, 1990.
- (DGW2007) Downey, Rod; Greenberg, Noam; Weber, Rebecca. Totally ω-computably enumerable degrees and bounding critical triples. J. Math. Log. 7 (2007), 145–171.

The notions "array nonrecursive" (now called array noncomputable or a.n.c. for short) and "not totally ω-c.e." were introduced in order to capture "multiple permitting" arguments.

In the first case the type of multiple permitting which is covered is explicitly described in the paper. In the second case we give such a description here.

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- In case of a permitting construction, x_e is enumerated into B only if at the same stage a number $y \leq x_e$ (or, more generally, a number $y < f(x_e)$ where f is a computable function) enters A ("A (f-)permits x_e").

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In this simple setting all noncomputable c.e. sets A eventually permit (i.e., permitting = noncomputable).

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    Then we say that A has to uniformly multiply permit.

What are the (uniformly) multiply permitting c.e. sets?

Array computable and totally ω-c.e. degrees

- A function $f(\cdot)$ is g-c.e. if there is a computable approximation $f(\cdot,\cdot)$ of $f(\cdot)$ such that (for all x)

$$|\{s : f(x, s+1) \neq f(x, s)\}| \leq g(x).$$

And f is ω-c.e. if f is g-c.e. for some computable function g.

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Multiple permitting vs. a.n.c. and not totally ω-c.e.

- Array noncomputable = multiple permitting (DJS1990)
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In order to show the first equivalence, we have to look at the original definition of an array noncomputable set in DJS1990 which was designed to capture this permitting notion.

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- A (total) v.s.a. $\mathcal{F} = \{F_n\}_{n \geq 0}$ is a (total) very strong array of intervals (v.s.a.i.) if F_n is an interval and $\max F_n < \min F_{n+1}$ $(n \geq 0)$.

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As shown in DJS1990, the a.n.c. degrees are closed upwards, contain all non-low$_2$-degrees and split the low and low$_2$-low degrees. (In fact all of these are true for the not totally ω-c.e. degrees too.)

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- In order to force up to $g(x_e)$ permittings one uses a "trigger set" V: whenever a further permitting is needed one enumerates a new element y from F_n into V. If A copies V on F_n this will force y to enter A later thereby granting the requested permitting.

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Though a.n.c. sets capture multiple permittings, it is somewhat awkward to work with this notion (due to the necessity of the trigger sets). Moreover, the a.n.c. sets are not wtt-invariant (in fact not even ibT-invariant) though the intuitive multiple permitting notion has this property. This led us to a closely related but somewhat more intuitive notion.

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Definition (Ambos-Spies)

Let $\mathcal{F} = \{F_n\}_{n \geq 0}$ be a v.s.a., let f be a strictly increasing computable function, let A be a c.e. set, and let $\{A_s\}_{s \geq 0}$ be a computable enumeration of A.

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- A is $\mathcal{F}$-permitting via f (and $\{A_s\}_{s \geq 0}$) if, for any partial computable function ψ,

$$\exists^\infty n \, \forall x \in F_n(\psi(x)\downarrow \Rightarrow A \restriction f(x) + 1 \neq A_{\psi(x)} \restriction f(x) + 1) \tag{1}$$

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A formal notion of multiply permitting c.e. sets: facts

Theorem (Ambos-Spies)

(i) *The wtt-degrees of the multiple permitting sets coincide with the wtt-degrees of the a.n.c. sets.*

(ii) *The multiple permitting property is wtt-invariant, in fact closed upwards under $\leq_{wtt}$. Moreover, for any c.e. splitting $A = A_0 \sqcup A_1$ of a multiple permitting set A, A_0 or A_1 is multiply permitting too.*

(iii) *If A is multiply permitting then A is $\mathcal{F}$-permitting for all v.s.a. $\mathcal{F}$ (but, in general, the corresponding permitting bound $f = f_{\mathcal{F}}$ depends on $\mathcal{F}$).*

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NB. The multiple permitting property is not T-invariant. Ambos-Spies and Monath have shown that there are c.e. Turing degrees **a** such that all c.e. sets in **a** are multiply permitting (hence wtt-equivalent to an a.n.c. set) but that such a degree **a** cannot be high.

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Some facts: The uniform multiple permitting property is wtt-invariant, in fact closed upwards under $\leq_{wtt}$. Moreover, for any c.e. splitting $A = A_0 \sqcup A_1$ of a u.m.p. set A, A_0 or A_1 is u.m.p. too.
The u.m.p. is not Turing-invariant. Moreover, (in contrast to m.p.) any c.e. Turing degree contains a c.e. set which is not u.m.p. (For this we show that h-simple sets are not u.m.p.)

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Theorem (Ambos-Spies and Losert)

A c.e. Turing degree is not totally ω-c.e. iff it contains a u.m.p. set.

So uniform multiple permitting (in the formal sense) characterizes the permitting power of not totally ω-c.e. c.e. degrees.

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We look at the following question: If a (finite) lattice $\mathcal{L}$ can be embedded into the partial ordering of the c.e. degrees $(\mathbf{R}, \leq)$, can it be embedded in any nontrivial principal ideal $\mathbf{R}(\leq \mathbf{a})$. If not, for what degrees $\mathbf{a}$ does such an embedding exist?

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$\mathcal{L} = \mathrm{N}_5$	$\mathbf{a} > \mathbf{0}$
$\mathcal{L} = \mathrm{M}_3$	$\mathbf{a}$ is not totally $< \omega^\omega$-c.e. (Downey and Greenberg ta)

An application: bounding lattice embeddings into the c.e. degrees

We look at the following question: If a (finite) lattice $\mathcal{L}$ can be embedded into the partial ordering of the c.e. degrees $(\mathbf{R}, \leq)$, can it be embedded in any nontrivial principal ideal $\mathbf{R}(\leq \mathbf{a})$. If not, for what degrees $\mathbf{a}$ does such an embedding exist?

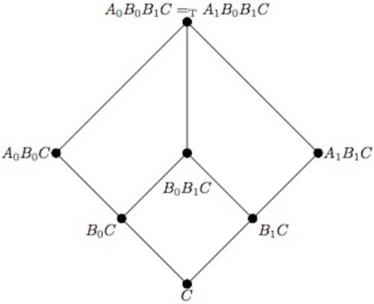
The finite lattice $\mathcal{L}$	is embeddable in $\mathbf{R}(\leq \mathbf{a})$ iff
$\mathcal{L}$ distributive	$\mathbf{a} > \mathbf{0}$
$\mathcal{L} = \mathrm{N}_5$	$\mathbf{a} > \mathbf{0}$
$\mathcal{L} = \mathrm{M}_3$	$\mathbf{a}$ is not totally $< \omega^\omega$-c.e. (Downey and Greenberg ta)
a critical triple	$\mathbf{a}$ is not totally ω-c.e. (Downey, Greenberg and Weber 2007)

(Here incomparable c.e. degrees $\mathbf{a_0}, \mathbf{a_1}, \mathbf{b}$ form a critical triple if $\mathbf{a_0} \vee \mathbf{b} = \mathbf{a_1} \vee \mathbf{b}$ and $\mathbf{a_0} \wedge \mathbf{a_1} \leq \mathbf{b}$.)

An application: bounding lattice embeddings (continued)

Theorem (Ambos-Spies and Losert)

If A is uniformly multiple permitting then the seven element meet-semidistributive but not join-semidistributive lattice S_7 can be embedded into $\mathbf{R}(\leq deg(A))$.



An application: bounding lattice embeddings (continued)

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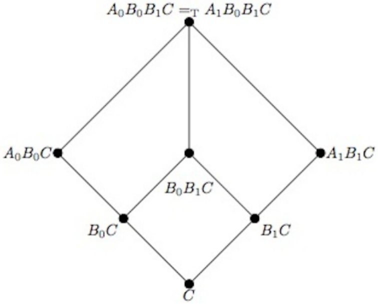

Corollary (Ambos-Spies and Losert)

The lattice S_7 can be embedded into $\mathbf{R}(\leq \mathbf{a})$ iff $\mathbf{a}$ is not totally ω-c.e.