On building models of Solovay theories

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The question and a first answer:

Fix a theory T.

Question

What do you need to know to compute (the atomic diagram of) a model of T?

Question

What do you need to know to compute all the models of T?

Necessary partial answer

For d to compute a model of T, we must have $T \cap \exists_n$ must be uniformly $\Sigma_n^0(d)$.

But is that enough?

Solovay theories

Definition

A theory is Solovay if $T \cap \exists_n$ is uniformly Σ^0_n.

Question

Which degrees compute a model/every model of Solovay theories?

Answer (Knight/Solovay)

A degree computes a model of every Solovay theory if and only if it computes a copy of a non-standard model of true arithmetic. In particular, the degree must compute every arithmetical degree.

No degree computes every model of every Solovay theory: Take true arithmetic, for example. It has $2^{\aleph_0}$ countable models. No degree can compute all of them.

Resurrecting the question

Suppose I tell you T is "nice".

Question

What do you need to know to compute all the models of the theory T?

For example, "nice" might mean $\aleph_0$-categorical, strongly minimal, $\aleph_1$-categorical, T has only countably many countable models, T is the theory of a free module over R for some (Noetherian/Artinian) ring R, etc.

If you are interested in a given class of theories, it should be a natural question to want to know what degrees can compute copies of all of its models.

Formalizing the question

Question

Which degrees can compute every model of a “nice” Solovay theory T?

Of course, the answer will depend on which instance of “nice” you pick. I think this is an important approach towards computable model theory, where “nice” is whatever class of theories you are already studying.

I can tell you something meaningful about the cases where “nice” is $\aleph_0$-categorical or where “nice” is strongly minimal.

An answer for $\aleph_0$-categorical theories

Theorem (Knight '94, improving Lerman-Schmerl '79)

If T is $\aleph_0$-categorical and Solovay, then $0'$ can compute a copy of every/the model of T.

Lerman and Schmerl constructed an example showing this is sharp as well. What do you actually need to know to build the model? The answer is clear from the proof. You need to be able to know what quantifier free formulas you can commit to on a tuple $\bar{a}\bar{b}$ while $\bar{a}$ satisfies a given $\forall_1$-formula. In other words, you need to be able to enumerate statements of the form $\exists\bar{x}\bar{y}\varphi^1(\bar{x}) \wedge \psi^0(\bar{x}, \bar{y})$, i.e. you need to enumerate $\exists_2$-facts about T.

That was a slight lie: You need $T \cap \exists_{n+1}$ to be $\Sigma^0_n(d)$ uniformly for each $n \geq 1$, and for the same reason, so I chose $n = 1$ to illustrate why.

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Strongly minimal theories

Definition

A first order theory T is *strongly minimal* if every definable subset of every model is a finite or co-finite subset of the model.

Examples

- A regular acyclic graph with finite valence (say, the theory of $\mathbb{Z}$ with the successor function);
- A vector space (say, the theory of $(\mathbb{Q}, +)$);
- An algebraically closed field, (say, the theory of $(\mathbb{C}, +, \cdot, 0, 1)$)

Algebraic closure and independence

Definition

For elements $\bar{a}, b \in M$, we say $b \in \operatorname{acl}(\bar{a})$ if there is a formula $\varphi(x, \bar{y})$ so that $\varphi(M, \bar{a})$ is finite and $M \models \varphi(b, \bar{a})$.
i.e., b is in a finite $\bar{a}$-definable set.

Definition

A set $S \subset M$ is independent if each $x \in S$ is not in $\operatorname{acl}(S \setminus \{x\})$.

Definition

The size of a maximal independent subset of M is the dimension of M.

Dimension characterizes models

In each of our examples, the notion of dimension characterizes models. This is not a coincidence.

Theorem (Baldwin-Lachlan)

If T is strongly minimal[a], then each model of T is determined by its dimension. If M is countable, then $\dim(M) \in \{0, 1, \ldots, \aleph_0\}$.

[a] They actually showed the result for $\aleph_1$-categorical theories, but I will talk only about strongly minimal theories

A first pass at connecting computing the theory to computing models

Results about how computing the theory and computing models are related:

Theorem (Harrington/Khissamiev '74)

If d computes an $\aleph_1$-categorical theory T, then d decidably presents every model of T.

Theorem (A. '13)

There is a strongly minimal theory T so that $T \equiv_T 0^{(\omega)}$ and every model of T is computable.

So, the right measure of computing the theory is not its full degree, but rather, we should assume T is Solovay.

Strongly minimal Solovay theories

Question (Our more refined question)

Which degrees compute every model of every strongly minimal Solovay theory?

Theorem (Khoussainov-Laskowski-Lempp-Solomon '07)

There is a strongly minimal theory where the prime model is computable and any presentation of a positive-dimensional model computes $0''$.

Theorem (A.-Knight)

If T is a strongly minimal Solovay theory, then $0'''$ can compute a copy of every model.

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Question left open in Andrews-Knight

Question (Left open in Andrews-Knight: 2 or 3?)

Does $0''$ present a copy of every model of a strongly minimal Solovay theory?

Does $0''$ present a copy of every model a strongly minimal theory with a computable model?

The answer: Neither, but closer to 3.

Theorem (A.-Lempp-Schweber)

A degree d can compute a copy of every model of a strongly minimal Solovay theory iff d is high over $0''$.

Theorem (A.-Lempp-Schweber)

A degree d can compute a copy of every model of a strongly minimal theory which has at least one computable model if and only if d is high over $0''$.

Clarification: d is high over $0''$ means: $d \geq_T 0''$ and $d' \geq_T 0^{(4)}$.

Subtle distinction: high over $0''$ vs $0'''$

Where does high over $0''$ come from? Where did $0'''$ come from?

Knight and I used a worker construction: The $0^{(n+3)}$-worker was tasked with building a n-quantifier presentation of a structure, which (at the n-quantifier level) looks like a model of T should look. Further, it has to build an isomorphism to the model built by the $0^{(n+4)}$-worker. In order to pass down the information needed in a finite injury fashion, the $0^{(n+4)}$-worker needs to determine not just $n+1$-quantifier statements, but a full $n+1$-quantifier type.

But when we ask for a computable presentation, that model on the bottom doesn't need to determine full 0-quantifier types, but rather just decide each atomic statement, one by one. This is precisely the reason that high over $0''$ suffices instead of $0'''$. If we demanded full 0-quantifier types, we would need $0'''$.

Hope for the future?

In model theory, the strongly minimal sets form building blocks that are used to understand everything from differential equations to complex manifolds to algebraic geometry.

We now know precisely what it takes to compute models of a strongly minimal theory. Where can we go with this?

One of the simplest settings where strongly minimal sets are used is the Baldwin-Lachlan proof of Morley's theorem: i.e. $\aleph_1$-categorical theories are built in a very simple way out of a strongly minimal theory.

Question (A sample test question)

If T is an $\aleph_1$-categorical Solovay theory, must all of its models be arithmetical? Is there an n so that all of its models are compuable from $0^{(n)}$?

Thank you

Thank you!