Randomness and uniform distribution modulo one

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Oberwolfach Workshop 1802 “Computability Theory”, January 7-13, 2018

How is randomness related to theory of uniform distribution?

Uniform distribution modulo one

For a real x, $\{x\} = x - \lfloor x \rfloor$.

Definition

A sequence of reals $(x_n)_{n\geq 1}$ is uniformly distributed modulo one, abbreviated u.d. mod 1, if for all $a, b \in [0,1]$,

$$\lim_{N\to\infty} \frac{\#\big\{n : 1 \leq n \leq N : \{x_n\} \in [a,b)\big\}}{N} = b - a$$

u.d. mod 1

Consider Lebesgue measure μ on $[0,1]$ and the product measure μ_∞ on $[0,1]^{\mathbb{N}}$.

Theorem (Hlawka, 1956)

μ_∞-almost all elements in $[0,1]^{\mathbb{N}}$ are u.d. in the unit interval.

Examples

Theorem (Bohl; Sierpinski; Weyl 1909-1910)

A real x is irrational if and only if $(nx)_{n\geq 1}$ is u.d. mod 1.

Theorem (Wall 1949)

A real x is Borel normal to base b if and only if $(b^n x)_{n\geq 1}$ is u.d. mod 1.

Martin-Löf randomness

Definition (Martin-Löf randomness 1965)

A real x is random if for every computable sequence $(V_n)_{n\geq 1}$ of computably enumerable open sets of reals such that $\mu(V_n) < 2^{-n}$,

$$x \notin \bigcap_{n\geq 1} V_n.$$

Koksma's General Metric Theorem

Definition (Koksma 1935)

Let $\mathcal{K}^{all}$ *be the class of sequences* $(u_n : [0,1] \to \mathbb{R})_{n\geq 1}$ *such that*

1. $u_n(x)$ *is continuously differentiable for every* n*,*
2. $u'_m(x) - u'_n(x)$ *is monotone on* x *for all* $m \neq n$*,*
3. *there exists* $K > 0$ *such that for all* $x \in [0,1]$ *and all* $m \neq n$*,* $|u'_m(x) - u'_n(x)| \geq K$*.*

Given a real x and $(u_n : [0,1] \to \mathbb{R})_{n\geq 1}$ consider $(u_n(x))_{n\geq 1}$.

Theorem (Koksma General Metric Theorem 1935)

Let $(u_n : [0,1] \to \mathbb{R})_{n\geq 1}$ *in* $\mathcal{K}^{all}$*. Then, for almost all (Lebesgue measure) reals* x *in* $[0,1]$*,* $(u_n(x))_{n\geq 1}$ *is u.d. mod* 1*.*

Avigad's Theorem

Theorem (Avigad 2013)

If a real x is Schnorr random then for every computable sequence $(a_n)_{n\geq 1}$ of distinct integers, $(a_n x)_{n\geq 1}$ is u.d. mod 1.

Effective Koksma class $\mathcal{K}$

Definition

Let $\mathcal{K}$ be the class of computable sequences $(u_n : [0,1] \to \mathbb{R})_{n\geq 1}$ in $\mathcal{K}^{all}$ such that the sequence of derivatives $(u'_n : [0,1] \to \mathbb{R})_{n\geq 1}$ is also computable.

Proper inclusion

Theorem 1

Let x be a real in $[0,1]$. If x is random then for every $(u_n : [0,1] \to \mathbb{R})_{n\geq 1}$ in $\mathcal{K}$ the sequence $(u_n(x))_{n\geq 1}$ is u.d. mod 1.

The proof considers $(u_n(x))_{n\geq 1}$ not u.d. mod 1 and constructs a Solovay test that is failed by x.

The converse of Theorem 1 does not hold.

Theorem 2

There is a real x in $[0,1]$ such that x is not random and for every $(u_n : [0,1] \to \mathbb{R})_{n\geq 1}$ in $\mathcal{K}$, $(u_n(x))_{n\geq 1}$ is u.d. mod 1.

Σ_1^0-u.d. mod 1

Definition

A sequence $(x_n)_{n\geq 1}$ of reals is Σ_1^0-u.d. mod 1 if for every Σ_1^0 set $A \subseteq [0,1]$,

$$\lim_{N\to\infty} \frac{1}{N} \# \Big\{ 1 \leq n \leq N : \{x_n\} \in A \Big\} = \mu(A).$$

Examples

Proposition

If x is computable and irrational then $(nx)_{n\geq 1}$ is u.d. mod 1 but not Σ^0_1 u.d mod 1.

Proposition (easy extension of Hlawka, 1956)

μ_∞-almost all elements in $[0,1]^{\mathbb{N}}$ are Σ^0_1-u.d. in the unit interval.

Inclusion

Theorem 3

Let x be a real number in $[0,1]$. If $(u_n : [0,1] \to \mathbb{R})_{n\geq 1}$ in $\mathcal{K}$ and $(u_n(x))_{n\geq 1}$ is Σ^0_1-u.d. mod 1 then x is random.

Characterization

Theorem (Franklin,Greenberg,Miller,Ng 2012; Bienvenu,Day,Hoyrup,Mezhirov,Shen 2012)

A real x is random if and only if $(2^n x)$ is Σ_1^0-u.d. mod 1.

Randomness and uniform distribution

for all $(u_n)_{n\geq 1}$ in $\mathcal{K}$, $(u_n(x))_{n\geq 1}$ is Σ^0_1-u.d. mod 1

$\Downarrow \quad \Uparrow?$

exists $(u_n)_{n\geq 1}$ in $\mathcal{K}$, $(u_n(x))_{n\geq 1}$ is Σ^0_1-u.d. mod 1

$\Downarrow? \quad \Uparrow$

$(2^n x)_{n\geq 1}$ is Σ^0_1-u.d. mod 1

$\Downarrow \quad \Uparrow$

x is random

$\Downarrow \quad \not\Uparrow$

for all $(u_n)_{n\geq 1}$ in $\mathcal{K}$ is $(u_n(x))_{n\geq 1}$ is u.d. mod 1

Discrepancy associated to random reals

Problem

Minimize the discrepancy of $(u_n(x))_{n\geq 1}$ *for* $(u_n : [0,1] \to \mathbb{R})_{n\geq 1}$ *in* $\mathcal{K}$ *and* x *random.*

Discrepancy associated random reals

Definition

$$D_N((x_n)_{n\geq 1}) = \sup_{0\leq u<v\leq 1} \left| \frac{\#\{n : 1 \leq n \leq N \textit{ and } u \leq \{x_n\} < v\}}{N} - (v-u) \right|$$

Thus, $(x_n)_{n\geq 1}$ is u.d. mod 1 if $\lim_{N\to\infty} D_N((x_n)_{n\geq 1}) = 0$.

Schmidt, 1972, proved that there is a constant C such that for *every* $(x_n)_{n\geq 1}$ there are infinitely many Ns with

$$D_N((x_n)_{n\geq 1}) \geq C\frac{\log N}{N}.$$

This lower bound is achieved by low-discrepancy sequences.

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