On low for speed sets

Laurent Bienvenu (CNRS & Université de Montpellier)
Rod Downey (Victoria University of Wellington)

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Lowness for speed

A recurring theme in computability theory:

Low$(\mathcal{N})$ $=$ set of oracles *X* such that relativizing the notion $\mathcal{N}$ to *X* leaves it unchanged.

- $\mathcal{N}$ $=$ halting set $\;\longrightarrow\;$ *Low*$(\mathcal{N})$ $=$ low
- $\mathcal{N}$ $=$ ML-random $\;\longrightarrow\;$ *Low*$(\mathcal{N})$ $=$ K-trivials
- $\mathcal{N}$ $=$ weakly 1-generic (or Kurtz random)
  $\longrightarrow\;$ *Low*$(\mathcal{N})$ $=$ non-dnr + hyperimmune-free

Lowness for speed

Allender proposed to study **lowness for speed:**

Definition (Allender)

X is low for speed (l.f.s) if every *decidable* set/language L that can be computed with oracle X in time f can be computed without oracle in time $poly(f)$.

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Does such an A exist? Obviously yes: take A to be in PTIME-computable! (note: X computable but EXPTIME-complete would not work, so lowness for speed is **not** closed under $\equiv_T$).

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Proof is a priority argument. One constructs A to be sparse, so that at stage t there are few candidates for $A \upharpoonright t$, thus for a functional Φ one can try to simulate all possible Φ^A in parallel (+ some very nice twist to handle Friedberg-Muchnik requirements).

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2. What is the situation outside c.e. sets? How big is the set LFS in terms of cardinality/category/measure? (category answered by Bayer and Slaman)
3. Closing under $\equiv_T$: what are the X such that equivalent to some low for speed? (note: every degree contains a non low for speed). Are such X closed downwards? under join?

Within c.e. sets

Can we characterize the c.e. sets in LFS? Seems very hard, but one can get partial results.

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There is a non-prompt c.e. set A such that A is not l.f.s., nor any $B \equiv_T A$.

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Theorem (BD)

If $A \geq_T \emptyset'$, then A is not l.f.s. (does not require A to be c.e.).

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Theorem (BD)

- There is a high c.e. set that is low for speed.
- A non-computable c.e. low set A cannot be low for speed(!)
- There is a non-computable low_2 c.e. set that is low for speed.

Outside the c.e. world

How common are low for speed sets? Can/should a generic be low for speed? How about randoms?

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So we might not know for a while whether LFS is meager or co-meager.

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However, LFS contains an homeomorphic copy of the 1-generics. Consider a doubly-exponentially sparse set S such as

$$S = \left\{2^{2^n} \mid n \in \mathbb{N}\right\}$$

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A fairly direct proof gives us:

Theorem (BD)
If G is 1-generic, then S_G is low for speed.

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Like for generics, one could expect a conditional behaviour of randoms w.r.t. lowness for speed, for example a dependance on the answer to $\mathrm{P} = \mathrm{BPP}$. This is not the case:

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Theorem (BD)
If A has DNC degree, it is not low for speed.

Proof inspired by Blum's speedup theorem.

Turing degrees and LFS

This last result also gives us that any $A \geq_T \emptyset'$ is not equivalent to any l.f.s. set. And from this:

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Proof: Take a 2-generic G_0 and consider $G_1 = G_0 \Delta \emptyset'$, also 2-generic. Both G_0 and G_1 are Turing equivalent to a l.f.s. set, but $G_0 \oplus G_1 \geq_T \emptyset'$ is not.

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The Turing degrees of LFS are not closed downwards.

Proof: extend the earlier result to show that a low c.e. *degree* does not contain any l.f.s. set. Take a non-computable c.e. set X which is l.f.s. and apply Sack's splitting theorem to get a low c.e. Y with $0 <_T Y <_T X$.

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How lowness for speed interacts with minimality is not fully solved, but we know at least:

Theorem (BD)
There exists a minimal Turing degree which does not contain any l.f.s. set.

(We do not know whether a l.f.s. set can be of minimal Turing degree)

Thank you!