Rogers Semilattices of Generalized computable numberings.

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Partially supported by Grant RFBR of 17.01.003247

A comprehensive and extensive study of generalized computable numberings was initiated at the end of the past century, through a unifying approach towards a notion of a computable numbering for a family of sets of constructive objects, suggested in the paper of S.Goncharov and A.Sorbi.

A program of such a study was outlined in the paper by S.A. Badaev, S.S. Goncharov. (*Theory of numberings: open problems.* In: Computability Theory and its Applications, P. Cholak, S. Lempp, M. Lerman and R. Shore eds.—Contemporary Mathematics, American Mathematical Society, 2000, vol. 257, pp. 23-38, Providence.) The first approach was connected with arithmetical hierarchy.

The series of papers was presented by S.Badaev, S.Goncharov, A.Sorbi, S.Podzorov. In these papers were studied the problem of structural properties of Rogers semilattices of families in classes of hierarchies of arithmetical and hyperarithmetical sets.

Generalization of computable numberings (Goncharov-Sorbi)

In the paper by S.S. Goncharov and A. Sorbi (*Generalized computable numberings and non-trivial Rogers semilattices.* Algebra and Logic, 1997, vol. 36, no. 6, pp. 359–369) was suggested some generalization of the notion of computable numberings.

let S^* be some family of objects with description in a formal language L. We fix some interpretation Int of the language L in S^*.

A numberings ν from the set of natural numbers N in the set we call computable, if there exist S множества S^* называем вычислимой, если существует вычислимая функция is из N в L такая, ч a computable function F such that $\nu(n) = int(f(n))$ for any $n \in N$.

We can consider some restriction on the classes of computable function from class F of function (polynomial, computable relative to oracle, or from some classes of hierarchies). We can fix this class F.

Generalization of computable numberings

We can define for subclass $S \subseteq S^*$ the set $\mathbf{R}(S, int, F)$ of all computable numberings relative to fix interpretation int for formal language L and class F of $computable$ functions. In the cases we have definable interpretation we will write without F, as $\mathbf{R}(S)$.

Generalization of computable numberings

If we have two F-computable numberings ν, μ from $\mathbf{R}(S, int, F)$ on S, we will say that ν F-reducible to μ ($\nu \leq_F \mu$, if there exists a function f from F such that $\nu(n) = \mu(f(n))$ for any $n \in N$.

In this case we have on $\mathbf{R}$ preorder $\leq_F$. Relative to this reducibility we have equivalence ($\nu =_F \mu$ if ($\nu \leq_F \mu$ and ($\mu \leq_F \nu$.

Let $R(S, int, F) \rightleftharpoons \mathbf{R}(S, int, F)/ =_F$ with partial order $\leq_F$.

The numbering $\nu \oplus \mu$ where $\nu \oplus \mu(2n) = \nu(n)$ and $\nu \oplus \mu(2n+1) = \mu(n)$ for any $n \in N$ is spermium of elements $\nu/ =_F$ and $\mu/ =_F$ in $(R(S, int, F), \leq_F)$.

This semilattice $(R(S, int, F), \leq_F)$ of F-computable numberings relative to interpretation int for L is called the Rogers semilattice.

Classical computable numberings

In classical numbering theory Yu. Ershov presented open problem about type of isomorphisms of Rogers semilattices on finite families of c.e. sets. S.Denisov constructed isomorphisms for finite families with smallest element and n another elements without inclusions.

If $R(S_0) \cong R(S_1)$ then $S_0' \cong S_1'$ where S' is the set of all essential elements of S with an order induced by the order $\subseteq$.

Open problems

Open problem. Is there a family c.e. sets with exactly 2 (n+2) minimal computable numberings? (Yu.Ershov)

Open problem. Let S, S' be finite families. In what cases $R(S, \Sigma^0_1)$ and $R(S', \Sigma^0_1)$ are isomorphic?

Computable numberings in the levels of arithmetical hierarchy

Theorem

There is lattices of $\Sigma^0_m(S_0)$-computable numbering of families of Σ^0_m-sets which is non-isomorphic any $R^0_n(S)$ of for $m > n+1$.

Theorem

The lattice of Σ^0_m-computable numbering of families of all Σ^0_m-sets is non-isomorphic any $R^0_n(S)$ of for $m > n+1$.

The question for Rogers semilattices R^0_n of arithmetical Σ^0_n-computable sets and R^0_m of Σ^0_m-computable numberings was solved by Badaev-Goncharov-Sorbi $(m > n+2)$) and Podzorov $(m > n+1)$.

Open problem. Is there a family S such that $R(S, \Sigma^0_{n+2})$ is not isomorphic to $R(S_0, \Sigma^0_{n+1})$ for any Σ_{n+1}-computable family S_0.

Arithmetical sets

Theorem

(D. Velegzhanina) There is infinitely many Friedberg arithmetical computable numberings of all arithmetical sets.

Theorem

(D. Velegzhanina) The Rogers semilattice of all arithmetical computable numberings of all arithmetical sets is not isomorphic to any Rogers semilattice of any Σ^0_n-computable family of Σ^0_n-computable numberings.

S.Badaev

Computable functionals(Yu. Ershov)

Theorem

(S.Ospichev) There is a Friedberg computable numbering of all τ-computable functionals sets.

Analytical hierarchy, James C. OWINGS, JR

Theorem

There is not a meta-c.e. numbering S(α)($\alpha < \omega_1$) Π^1_1-sets such that it is Friedberg numbering of all Π^1_1-sets.

Theorem

There is not a Σ^1_1-numbering Σ^1_1-sets such that it is a Friedberg numbering of all Σ^1_1-sets for $1 \le n \le 2$.

Theorem

There is not a Σ^1_n-numbering Σ^1_n-sets such that it is Friedberg numbering of all Σ^1_n-sets for $1 \leq n \leq 2$.

Proof is without metarecursion.

Open problem. Is it true that there is not a Σ^1_n-numbering Σ^1_n-sets such that it is Friedberg numbering of all Σ^1_n-sets for $n \geq 3$.

Computable numberings in classical case

Theorem (Khutoretskii's Theorem)

Let S be a family of computable enumerable sets.

1. *If $\mu \not\leq \nu$ are computable numberings of S then there is a computable numbering π of S with $\pi \not\leq \nu$ and $\mu \not\leq \pi \oplus \nu$.*
2. *If the Rogers semilattice $R^0_1(S)$ of S contains more than one element, then it is infinite.*

Badaev-Lempp result in Computable numberings in Ershov hierarchy

Theorem (Badaev S.A. and Steffen Lempp, A decomposition of the Rogers semilattice of a family of d.c.e. sets. The Journal of Symbolic Logic, Vol. 74, No. 2, 2009)

There is a family $\mathcal{F}$ of d.c.e. sets, and there are computable numberings μ and ν of the family $\mathcal{F}$ such that for any computable numbering π of $\mathcal{F}$, either $\mu \leq \pi$ or $\pi \leq \nu$. In addition, we can ensure the following:

- *$\mathcal{F}$ is a family of c.e. sets and ν is a computable numbering of $\mathcal{F}$ as a family of c.e. sets;*
- *both μ and ν can be made Friedberg and thus minimal numberings; and so*
- *any computable numbering π of $\mathcal{F}$ satisfies $\pi \equiv \nu$ or $\mu \leq \pi$.*

Ershov hierarchy and principal and minimal computable numberings

S.Ospichev, Infinite family of Σ_a^{-1}-sets with a unique computable numbering, Journal of Mathematical Sciences, 2013, 188:4, 449Ц451

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Ershov hierarchy and principal and minimal computable numberings

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Badaev S.A., Manat M., Andrea Sorbi, Friedberg numberings in the Ershov hierarchy, Arch. Math. Logic, 2015.

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One open question in Ershov hierarchy

It is interesting questions about computable numberings relative to classes of Ershov Hierarchy.

Open problem. Is there of a family of Σ_2^{-1}-множеств с точно двумя Σ_2^{-1}-computable numberings.

Ershov hierarchy

Together with A.Sorbi and S. Badaev we proved the next result

Theorem

There exist the infinite family S *of* Σ_2^{-1} *sets with Rogers semilattece* $R_2^{-1}(S)$ *of all* Σ_2^{-1}*(-numberings of the family* S *with two elements* 0 *and* 1 *such that* $0 < 1$ *and (*$c = 0$ *or* $1 \leq c$*) for any* c.

We can use the paper "Reductions between Types of Numberings"by Ian Herbert, Mustafa Manat and Frank Stephan.

Corollary. There exist the infinite family S of Σ_n^{-1} sets with Rogers semilattes $R_n^{-1}(S)$ of all Σ_n^{-1}(-numberings of the family S with two elements 0 and 1 such that $0 < 1$ and $(c = 0$ or $1 \leq c)$ for any c. .