Weihrauch reducibility, highness classes, cardinal characteristics, forcing

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Cardinal characteristics

In light of the independence of CH, set theorists tried to look at variants of the question “how many real numbers are there?”
For example:

- How many null sets does it take to cover the real line?
- How many functions does it take to dominate all functions $f : \omega \to \omega$?

Because of independence, the meaningful question is: how do these potentially different cardinalities relate to each other, i.e.: what’s provable in ZFC?
For example:

- If κ many functions suffice to dominate all functions, then κ many meagre sets suffice to cover the real line.

The Vojtas template

Many cardinal characteristics are of the form: the smallest number of solutions required to solve all instances.

Definition

Let A be a binary relation: $A \subseteq A_{\mathtt{inst}} \times A_{\mathtt{sol}}$.

$$\mathrm{Card}(A) = \min\{|Z| \,:\, (\forall x \in A_{\mathtt{inst}})(\exists z \in Z) \,:\, xAz\}.$$

For example:

- $\mathtt{Dom}$: the domination relation between functions; $\mathrm{Card}(\mathtt{Dom}) = \mathfrak{d}$.
- $\mathtt{Capture}(\mathscr{M})$: an instance is $x \in \mathbb{R}$; a solution is a meagre set $A \ni x$.
  $\mathrm{Card}(\mathtt{Capture}(\mathscr{M})) = \mathbf{cov}(\mathscr{M})$.

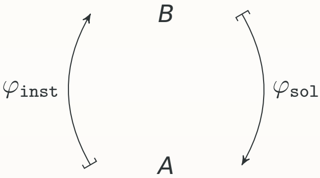
Duality

Every Weihrauch problem has a dual: $yR^{\perp}x$ iff $\neg(xRy)$.
For our examples:

- $\mathtt{Dom}^{\perp} = \mathtt{Esc}$, the problem of finding a function escaping a given function.
  $\text{Card}(\mathtt{Esc}) = \mathfrak{b}$, the unbounding number.
- $\mathtt{Capture}(\mathscr{M})^{\perp} = \mathtt{Pass}(\mathscr{M})$, the problem of finding a point outside a given meagre set.
  $\text{Card}(\mathtt{Pass}(\mathscr{M})) = \textbf{non}(\mathscr{M})$, the smallest size of a non-meagre set.

Morphisms

Many ZFC-proofs of inequalities between cardinals are obtained by morphisms between relations (a.k.a. Weihrauch problems).



- If there is a morphism from A to B then $\mathrm{Card}(A) \leqslant \mathrm{Card}(B)$.
- If $A \to B$ then $B^{\perp} \to A^{\perp}$.

Simple examples of morphisms

- $\mathtt{Esc} \to \mathtt{Dom}$: map an instance to itself; a solution g to $g+1$.

As a result: $\mathfrak{b} \leqslant \mathfrak{d}$.

- $\mathtt{Capture}(\mathscr{M}) \to \mathtt{Dom}$: map an instance to itself; a solution g to the set of functions dominated by g.
  It follows that $\mathtt{Esc} \to \mathtt{Pass}(\mathscr{M})$.

As a result: $\mathbf{cov}(\mathscr{M}) \leqslant \mathfrak{d}$ and $\mathfrak{b} \leqslant \mathbf{non}(\mathscr{M})$.

Highness classes

In his thesis, Rupprecht used the Vojtas template to define often familiar notions of oracular strength. (See also Brendle, Brooke-Taylor, Nies, Ng.)

Definition

For a Weihrauch problem A, we let $\mathrm{H}(A)$ be the set of oracles $x \in 2^\omega$ which compute a solution $y \in A_{\mathtt{sol}}$ that solves all computable instances in $A_{\mathtt{inst}}$.

- $\mathrm{H}(\mathtt{Dom})$ is high;
- $\mathrm{H}(\mathtt{Esc})$ is hyperimmune;
- $\mathrm{H}(\mathtt{Pass}(\mathscr{M}))$ is computing a weakly 1-generic;
- $\mathrm{H}(\mathtt{Capture}(\mathscr{M}))$ is computing a meagre set containing all computable reals (weakly meagre englufing).

Computable morphisms

We now restrict ourselves to computable morphisms (though we allow nonuniformity).

Proposition (Rupprecht)

If $A \to B$ then $\mathsf{H}(B) \to \mathsf{H}(A)$.

- $\mathtt{Esc} \to \mathtt{Dom}$: high implies hyperimmune.
- $\mathtt{Capture}(\mathscr{M}) \to \mathtt{Dom}$: High implies weakly meagre engulfing.
- $\mathtt{Esc} \to \mathtt{Pass}(\mathscr{M})$: computing a weakly 1-generic implies hyperimmune.

To get the arrows right, we let

$$\mathsf{NL}(A) = \mathsf{H}(A^{\perp}).$$

Example: lowness for Schnorr tests

- **cof**$(\mathscr{N})$ is the smallest size of a set of traces which trace every function (Bartoszyński 1984). This arises from a morphism equivalence between $\texttt{Cover}(\mathscr{N})$ and $\texttt{Trace}$.

  As a result: lowness for Schnorr tests is equivalent to computable traceability (Terwijn Zambella 2001).

Example: The Γ problem

For $x, y \in 2^\omega$, let

$$d(x,y) = \limsup_n d_H(x\upharpoonright_n, y\upharpoonright_n).$$

- For $p \in [0,1]$, $x \, \mathtt{Far}(p) \, y$ means $d(x,y) \geqslant p$.

  $x \in \mathrm{H}(\mathtt{Far}(p))$ implies $\Gamma(x) \leqslant 1 - p$.

  For $p, q \in (1/2, 1)$, $\mathrm{H}(\mathtt{Far}(p)) = \mathrm{H}(\mathtt{Far}(q))$ (Monin); as a result, $\Gamma(x) < 1/2 \Rightarrow \Gamma(x) = 0$.

Except... that we don't quite get morphism equivalence.

Operations on Weihrauch problems

For Weihrauch problems A and B, define the problem $A \times B$: an instance is a pair of instances $(a, b) \in A_{\mathtt{inst}} \times B_{\mathtt{inst}}$; a solution is $(c, d) \in A_{\mathtt{sol}} \times B_{\mathtt{sol}}$ such that aAc and bBd.

Proposition

- $\mathsf{Card}(A \times B) = \max\{\mathsf{Card}(A), \mathsf{Card}(B)\}$.
- $\mathsf{H}(A \times B) = \mathsf{H}(A) \cap \mathsf{H}(B)$.

The dual $A + B$ replaces and with or.

The morphisms we get are between sums of finitely many copies of $\mathtt{Far}(p)$.

Example: lowness for meagre sets

The most useful operation is sequential composition $A * B$ (Blass / Brattka, Gherardi, Marcone).

- $\text{Card}(A * B) = \max\{\text{Card}(A), \text{Card}(B)\}$;
- $\text{NL}(A * B) = \text{NL}(A) \cup \text{NL}(B)$.

$$\texttt{Cover}(\mathscr{M}) \to \texttt{Pass}(\mathscr{M}) * \texttt{Dom}.$$

As a result:

- $\mathbf{cof}(\mathscr{M}) = \max\{\mathfrak{d}, \mathbf{non}(\mathscr{M})\}$;
- Non-lowness for meagre sets is equivalent to hyperimmue or DNR (Stephan, Yu).

Other uses for sequential composition:

- Lowness for Kurtz tests (Greenberg, J. Miller).
- i.o.e. functions and weak meagre engulfing.

Generalise the question

Definition

- For a Turing ideal I, $x \in \mathrm{H}^I(A)$ if x computes (mod I) a solution for all instances in I.
- (Kihara) $x \in \mathrm{H}^{\Delta^1_1}(A)$ if some $y \in \Delta^1_1(x)$ solves all Δ^1_1 instances.

If $A \to B$ then implication holds in all settings: $\mathrm{H}^I(B) \to \mathrm{H}^I(A)$ and $\mathrm{H}^{\Delta^1_1}(B) \to \mathrm{H}^{\Delta^1_1}(A)$. On the other hand, ideals with closure properties often allow for more separations.

- $\mathtt{Pass}(\mathscr{M})$ vs. $\mathtt{Esc}$: are not equivalent for Δ^1_1 (Kihara).

Problem

Characterise the ideals I for which $\mathrm{H}^I(\mathtt{Pass}(\mathscr{M})) = \mathrm{H}^I(\mathtt{Esc})$.

Forcing

The standard way to show that $\mathrm{Card}(A) \leqslant \mathrm{Card}(B)$ is not provable in ZFC is to iterate forcing that adds a real in $\mathrm{NL}^V(A)$ but no real in $\mathrm{NL}^V(B)$.

Metatheorem

If $\mathrm{Card}(B) < \mathrm{Card}(A)$ *is consitent then for some ideal* I*,* $\mathrm{NL}^I(A) \twoheadrightarrow \mathrm{NL}^I(B)$*.*

Indeed, take any $I = 2^\omega \cap M$ where M is transitive and models ZFC.

Forcing

- If $I \models \mathtt{ATR}_0$ then Laver forcing "works" over I; as a result, there is an I-dominating function which is not I-strongly meagre engulfing.
- If $I \models \mathtt{ATR}_0$ then Hechler forcing "works" over I; as a result, there is an I-strongly meagre engulfing real which is not I-strongly null engulfing.

Blass-Shelah forcing

- $x \, \mathtt{Split} \, y$ means that y splits x: $x \cap y$ and $x \cap y^\complement$ are both infinite.
- $\mathrm{Card}(\mathtt{Split}) = \mathfrak{s}$ is the "splitting number".
- $\mathrm{NL}(\mathtt{Split})$ is computing an r-cohesive set (a set not split by any computable set).

Blass-Shelah forcing can be used to show the consistency of $\mathfrak{b} < \mathfrak{s}$.
It adds an unsplittable set without adding a dominating function.
Blass-Shelah forcing "works" over models of $\mathtt{WKL}_0$.
Using the existence of a HIF Scott set:

Theorem (Jockusch,Stephan)

There is an r-cohesive set which is not high.

Thank you.