When is a property expressed in infinitary logic also pseudo-elementary?

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This is joint work with Barbara Csima and Nancy Day.

Definition

A class $\mathbb{K}$ of $\mathcal{L}$-structures is a PC-class (pseudo-elementary class) if there is a language $\mathcal{L}^* \supseteq \mathcal{L}$ and an elementary first-order sentence ϕ such that

$$\mathbb{K} = \{\mathcal{M} \mid \text{there is an } \mathcal{L}^*\text{-structure } \mathcal{M}^* \text{ expanding } \mathcal{M} \text{ with } \mathcal{M}^* \vDash \phi\}.$$

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Example

The class of disconnected graphs is a PC-class.

A graph G is not connected iff
there is a transitive $S \supseteq E$ and a, b such that $(a, b) \notin S$.

Definition

The $\mathcal{L}_{\omega_1\omega}$-formulas are built up inductively as follows:

- atomic formulas
- $\neg\varphi$, where φ is an $\mathcal{L}_{\omega_1\omega}$-formula
- $(\exists x)\varphi$, where φ is an $\mathcal{L}_{\omega_1\omega}$-formula
- $(\forall x)\varphi$, where φ is an $\mathcal{L}_{\omega_1\omega}$-formula
- if $(\varphi_i)_{i\in\omega}$ are $\mathcal{L}_{\omega_1\omega}$-formulas, then so is $\bigwedge\!\!\!\bigwedge_{i\in\omega}\varphi_i$
- if $(\varphi_i)_{i\in\omega}$ are $\mathcal{L}_{\omega_1\omega}$-formulas, then so is $\bigvee\!\!\!\bigvee_{i\in\omega}\varphi_i$

Example

The class of disconnected graphs is also defined by the $\mathcal{L}_{\omega_1\omega}$ sentence:

$$\exists x_1, x_2 \bigwedge\!\!\!\bigwedge_{n\in\mathbb{N}} \forall y_1, \dots, y_n \quad \neg(x_1 E y_1 \wedge y_1 E y_2 \wedge \cdots \wedge y_n E x_2).$$

Example

Let ϕ be a first-order sentence. The class $\mathbb{K}$ of infinite models of ϕ is a PC-class and $\mathcal{L}_{\omega_1\omega}$-definable.

$\mathcal{A} \models \phi$ is infinite if and only if there is a linear order $\leq$ on $\mathcal{A}$ such that $(\forall x)(\exists y)[y \succ x]$.

$\mathbb{K}$ also defined by the infinitary sentence

$$\phi \wedge \bigwedge\!\!\!\bigwedge_{n \in \mathbb{N}} (\exists x_0, \ldots, x_n) \left[\bigwedge_{i \neq j} x_i \neq x_j \right].$$

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Example

The class of non-well-founded linear orders is a PC-class.

It is not $\mathcal{L}_{\omega_1\omega}$-definable.

We have two extensions of elementary first-order logic in two different directions. It is natural to ask what these two extensions have in common:

Question

Characterize the classes which are both pseudo-elementary class and definable by an infinitary sentence.

There are actually four variants of pseudo-elementary classes:

- PC
- PC′
- PC_Δ
- PC'_Δ

The Δ means that we are allowed a theory (rather than a sentence) and the ′ means that we are allowed to add new sorts.

The classes with ′ are a little difficult to define. In any case:

Theorem (Makkai)

PC_Δ *and* PC'_Δ *are the same.*

Definition

A class $\mathbb{K}$ of $\mathcal{L}$-structures is a PC_Δ-class if there is a language $\mathcal{L}^* \supseteq \mathcal{L}$ and an elementary first-order theory T such that

$$\mathbb{K} = \{\mathcal{M} \mid \text{there is an } \mathcal{L}^*\text{-structure } \mathcal{M}^* \text{ expanding } \mathcal{M} \text{ with } \mathcal{M}^* \vDash T\}.$$

We want to know: Which classes which are both PC_Δ and $\mathcal{L}_{\omega_1\omega}$-elementary?

We have two examples so far of classes which are in this intersection:

Disconnected graphs, defined by:

$$\exists x_1, x_2 \bigwedge\!\!\!\bigwedge_{n\in\mathbb{N}} \forall y_1, \ldots, y_n \quad \neg(x_1 E y_1 \wedge y_1 E y_2 \wedge \cdots \wedge y_n E x_2).$$

Infinite models of a first-order sentence ϕ, defined by:

$$\phi \wedge \bigwedge\!\!\!\bigwedge_{n\in\mathbb{N}} (\exists x_0, \ldots, x_n) \left[\bigwedge_{i\neq j} x_i \neq x_j \right].$$

Definition

An $\mathcal{L}_{\omega_1\omega}$-sentence φ is a conjunctive formula if it can be written in normal form without any infinite disjunctions.

More concretely, the conjunctive formulas are defined inductively as follows:

- every finitary quantifier-free sentence is a conjunctive formula
- if φ is a conjunctive formula, then so are $(\exists x)\varphi$ and $(\forall x)\varphi$
- if $(\varphi_i)_{i\in\omega}$ are conjunctive formulas, then so is $\bigwedge\!\!\!\bigwedge_{i\in\omega}\varphi_i$.

Theorem

Let $\mathbb{K}$ be a class of structures. The following are equivalent:

- $\mathbb{K}$ *is both a* $\mathsf{PC}_{\boldsymbol{\Delta}}$*-class and* $\mathcal{L}_{\omega_1\omega}$*-elementary.*
- $\mathbb{K}$ *is defined by a conjunctive sentence.*

Lemma (Modified Interpolation Theorem)

Suppose ϕ_1 is a conjunctive sentence and ϕ_2 is an $\mathcal{L}_{\omega_1\omega}$-sentence with $\phi_1 \vDash \phi_2$. These sentences may be in different languages.

There is a conjunctive sentence θ such that $\phi_1 \vDash \theta$, $\theta \vDash \phi_2$, and every relation, function and constant symbol occurring in θ occurs in both ϕ_1 and ϕ_2.

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Corollary

Let $\mathbb{K}$ be a class of $\mathcal{L}$-structures closed under isomorphism. If $\mathbb{K}$ is both a PC_Δ-class and $\mathcal{L}_{\omega_1\omega}$-elementary, then it is defined by a conjunctive sentence.

Theorem

Let $\mathbb{K}$ be a class definable by a computable conjunctive sentence in a finite language.

Then $\mathbb{K}$ is a PC' *class.*

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Relativizing, and using the fact that $\mathrm{PC}'_{\Delta} = \mathrm{PC}_{\Delta}$,

Corollary

Let $\mathbb{K}$ *be a class definable by a conjunctive sentence.*

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Theorem

Let $\mathbb{K}$ be a class definable by a computable conjunctive sentence in a finite language.

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Corollary

Let $\mathbb{K}$ be a class definable by a conjunctive sentence.

Then $\mathbb{K}$ is a PC_{Δ} class.

Question

If $\mathbb{K}$ is both a PC-class and $\mathcal{L}_{\omega_1\omega}$-elementary, then is it defined by a computable conjunctive sentence?

Example

The class of graphs with no cycles of prime length is a PC'-class.

Example

The class of graphs with at least one cycle of length p for each prime p is a PC'-class.

Theorem (Mal'tsev, Tarski)

If $\mathbb{K}$ *is a* PC'_{Δ}*-class which is closed under substructures, then it axiomatized by a set of universal sentences.*

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Theorem

Let $\mathbb{K}$ be a class of structures. The following are equivalent:

- $\mathbb{K}$ *is a PC'-class which is closed under substructures,*
- $\mathbb{K}$ *is axiomatized by a computable universal theory.*

Workshop on Computability Theory and its Applications

June 4-8, 2018

University of Waterloo, ON, Canada

Organizing Committee: Laurent Bienvenu, Peter Cholak, Barbara Csima, Matthew Harrison-Trainor.

Invited plenary speakers: Damir Dzhafarov, Bjørn Kjos-Hanssen, Joseph Miller, Selwyn Ng, Jan Reimann, Richard Shore, Linda Brown Westrick.

Public lecture: Antonio Montalbán.

For more information, visit the website,:

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