Characterizing the continuous degrees

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- Our main characterization captures the continuous degrees using a simple structural property.
- From this it follows that the continuous degrees are first-order definable in the partial order of the enumeration degrees.

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Definition. $A \leqslant_e B$ if there is a c.e. set W such that

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Proposition. The embedding $\iota \colon \mathcal{D}_T \to \mathcal{D}_e$, defined by

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It is easy to see that there are nontotal enumeration degrees. In fact, a sufficiently generic or random $A \subseteq \omega$ has nontotal degree.

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- The binary expansion of a real x is computable from every name. (But this is nonuniform because of the dyadic rationals!)
- The binary expansion of x computes a name for x.
- This is the least Turing degree name for x; it is natural to take this as the *Turing degree* of x.

Computable metric spaces

Definition. A *computable metric space* is a metric space $\mathcal{M}$ together with a countable dense sequence $Q^{\mathcal{M}} = \{q_n^{\mathcal{M}}\}_{n\in\omega}$ on which the metric is computable (as a function $\omega^2 \to \mathbb{R}$).

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Example. The *Hilbert cube* is $[0,1]^\omega$ with the metric

$$d(\alpha, \beta) = \sum_{n\in\omega} |\alpha(n) - \beta(n)|/2^n.$$

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As before, names let us transfer computability-theoretic notions to computable metric space.

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This reducibility induces the *continuous degrees*.

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But why are there nontotal continuous degrees?

Nontotal continuous degrees: quick proof

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- If $x \in [0,1]^\omega$ has total degree, then there is a $y \in 2^\omega$ and Turing functionals Γ, Ψ that map (names of) x to (names of) y and back.

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- The Hilbert cube $[0,1]^\omega$ is strongly infinite dimensional, hence not a countable union of zero dimensional subspaces.
- So some $x \in [0,1]^\omega$ is not covered by one of these patches. $\square$

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So we have another proof that nontotal continuous degrees exist.

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Theorem (Eilenberg and Montgomery 1946). Assume that $\Psi\colon [0,1]^\omega \to [0,1]^\omega$ is a multivalued function with closed graph such that $\Psi(\alpha)$ is nonempty and convex for each $\alpha \in [0,1]^\omega$. Then Ψ has a fixed point α (i.e., $\alpha \in \Psi(\alpha)$).

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So a total degree $\mathbf{a}$ is PA if and only if it bounds a nontotal continuous degree. Relativizing this fact we obtain:

Theorem (M. 2004). Let $\mathbf{b} \leqslant \mathbf{a}$ be total. There is a nontotal continuous degree $\mathbf{c} \in (\mathbf{b}, \mathbf{a})$ if and only if $\mathbf{a}$ is PA relative to $\mathbf{b}$.

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There are nontotal continuous degrees, so there are nontotal almost total degrees. This is the only way we know how to produce nontotal almost total degrees. (In particular, we have no "direct" construction.)

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Aside. We can also define a uniform version of almost totality. It is not too difficult to prove:

Theorem (AIMS). An enumeration degree is uniformly almost total if and only if it is continuous.

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3. For every $a \in A$ and $\sigma \in 2^{<\omega}$, there is a $\tau \succeq \sigma$ such that the range of $\Delta(A \oplus \tau \oplus \overline{\tau})$ contains a.

(Uniform) Codability

Definition. Let $A \subseteq \omega$. Call $U \subseteq 2^\omega$ a $\Sigma^0_1\langle A\rangle$ *class* if there is a set of strings $W \leqslant_e A$, such that

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- (AIMS) Codability implies uniform codability.

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- Therefore, A is uniformly codable.

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- We think of clopen sets $C \subseteq 2^\omega$ such that $(\forall X \in C)\ D \subseteq W^X$ as *potential witnesses* that $D \subseteq A$.

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- If $D \nsubseteq A$, then for any clopen $C \subseteq 2^\omega$ s.t. $(\forall X \in C)\ D \subseteq W^X$, it will be the case that $P \cap C = \varnothing$. This is also $\Sigma^0_1\langle A\rangle$.

- We think of clopen sets $C \subseteq 2^\omega$ such that $(\forall X \in C)\ D \subseteq W^X$ as *potential witnesses* that $D \subseteq A$.

- If $D \subseteq A$, then *at least one* witness is verified (positively from an enumeration of A). If $D \nsubseteq A$, then *all* witnesses are refuted $(\ldots)$.

Exploiting uniform codability

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- If $D \subseteq A$, then *at least one* witness is verified (positively from an enumeration of A). If $D \nsubseteq A$, then *all* witnesses are refuted (...).
- Iterating this observation, we get the notion of *holistic sets*.

Holistic sets

Definition. $S \subseteq \omega^{<\omega}$ is *holistic* if for every $\sigma \in \omega^{<\omega}$,

1. $(\forall n)\ \sigma^\frown(2n)$ and $\sigma^\frown(2n+1)$ are not both in S,
2. If $\sigma \in S$, then $(\exists n)\ \sigma^\frown(2n+1) \in S$.
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Lemma (AIMS). If $A \subseteq \omega$ is uniformly codable, then there is a holistic set $S \equiv_e A$.

We don't need it, but it is easy to show:

Proposition (AIMS). Every holistic set is uniformly codable.

The holistic space

Definition. Let

$$\mathcal{H} = \{S \subseteq \omega^{<\omega} \colon S \text{ is holistic}\}.$$

For each $\sigma \in \omega^{<\omega}$, let $O_\sigma = \{S \in \mathcal{H} \colon \sigma \in S\}$. These sets form a subbasis for the desired topology, i.e., their finite intersections form a basis. We call the resulting topological space the *holistic space*.

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The following is straightforward:

Fact (AIMS). $\mathcal{H}$ is second countable, Hausdorff, and regular.

Therefore, $\mathcal{H}$ satisfies the hypotheses of Urysohn's metrization theorem (1925–1926), so:

Fact (AIMS). $\mathcal{H}$ is metrizable.

Effective Urysohn's theorem

Theorem (Schröder 1998). Let $\mathcal{X}$ be a computable topological space (which implies second countable). If $\mathcal{X}$ is Hausdorff and computably regular, then there is a computable metric on $\mathcal{X}$ that generates the original topology.

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Moreover, it is easy to produce a computable dense set of point in $\mathcal{H}$. Therefore:

Lemma (AIMS). $(\mathcal{H}, d)$ is a computable metric space.

Finally, we can show:

Lemma (AIMS). If $S \in \mathcal{H}$, then the continuous degree of S as a point in $(\mathcal{H}, d)$ is the same as the enumeration degree of S.

The main theorem

Putting it all together:

Theorem (AIMS)

Let **a** be an enumeration degree. The following are equivalent:

1. **a** is (uniformly) almost total,

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4. $\mathbf{a}$ is continuous.

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Theorem (Cai, Ganchev, Lempp, M., and Soskova 2016).
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It is not known to be definable in the Turing degrees.

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  - Conversely, the fact that the Hilbert cube is not a countable union of subspaces of Cantor space follows easily from the fact that there is a nontotal continuous degrees in every cone.

So a purely topological fact is reflected in the structure of the enumeration degrees.

Thank you!